Extending the Hajnal-Szemerédi Theorem to directed graphs

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Background - perfect tilings

Definition (Tiling)

A *tiling* is a collection of vertex disjoint subgraphs called *tiles*.

Definition (Perfect tiling)

A *perfect tiling* or *factor* is *tiling* that spans the vertex set.

Theorem (Simple example)

If G is a graph on $2k$ vertices and $\delta(G) \geq k$ then G contains k independent edges, that is, a perfect K_2-tiling.

Background - perfect tilings with cliques

Definition (triangle)

Call a K_3 a *triangle*.

Theorem (Corrádi,Hajnal 1964)

If G is a graph on $3k$ vertices and $\delta(G) \geq 2k$ then G contains k independent triangles.

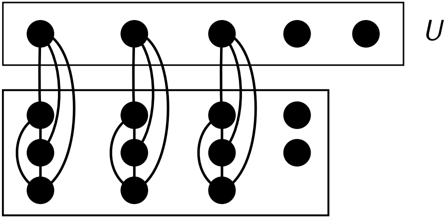
Theorem (Hajnal,Szemerédi 1970)

If G is a graph on sk vertices and $\delta(G) \geq (s-1)k$ then G contains k independent K_s.

Hajnal-Szemerédi is tight

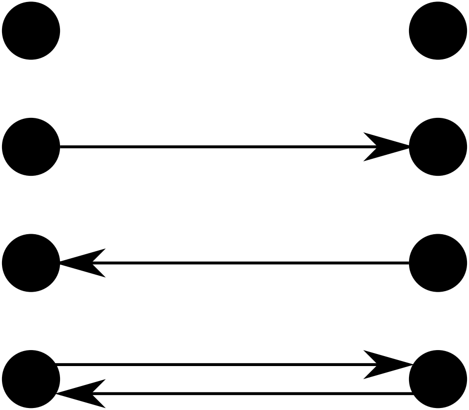
Example

- Consider the graph G on sk vertices with an independent set U of order $k+1$ and all other possible edges.
- Note that $d(u) = sk - (k+1) = (s-1)k - 1$ for $u \in U$.
- G cannot contain k independent K_s since a K_s can have at most one vertex in U.



(Simple) digraphs

Between any two vertices, there is at most one edge in either direction and no loops.



Two notions of degree for digraphs

Let D be a digraph.

Definition (Total degree)

For any $v \in V(D)$ let $d_t(v) := d^+(v) + d^-(v)$ and let $\delta_t(D) := \min\{d_t(v) : v \in V(D)\}$

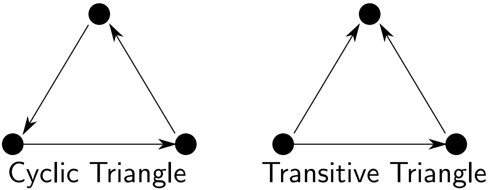
Definition (Semi-degree)

For any $v \in V(D)$ let $d_t(v) := \min\{d^+(v), d^-(v)\}$ and let $\delta_0(D) := \min\{d_0(v) : v \in V(D)\}$

Triangle factors in digraphs

Theorem (Czygrinow,Kierstead,M. 2012+ (Triangle factor))

If D is a digraph on $3k$ and $\delta_t(D) \geq 4k-1$ then D has a perfect tiling consisting of t transitive triangles and c cyclic triangles where $t+c=k$ and $t \geq 1$



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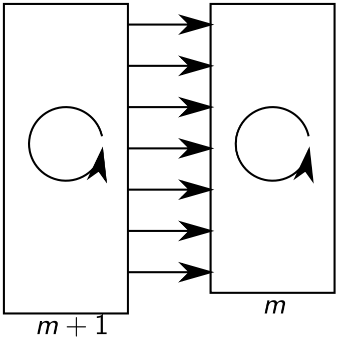
- The theorem is tight for the same reason Corrádi-Hajnal is tight. That is, if there is an independent set of size $k + 1$, D does not have a triangle factor.
- It is also possible to construct digraphs that meet the degree condition but do not have a cyclic triangle factor.

Triangle factors is tight

Example (Wang)

If $3k = 2m + 1$, the following digraph D has minimum total degree $2(m-1) + m + 1 = 3m - 1 = \frac{3}{2}(3k-1) - 1$ and no cyclic triangle factor.



Triangle factors - conjectures

Theorem (Triangle factor)

If D is a digraph on $3k$ vertices and $\delta_t(D) \geq 4k - 1$ then D has a perfect tiling consisting of t transitive triangles and c cyclic triangles where $t + c = k$ and $t \geq 1$

Theorem (Wang 2000)

If D is a digraph on $3k$ vertices and $\delta_t(D) \geq \frac{3}{2}(3k - 1)$ then D contains k independent cyclic triangles.

Problem

If we require D to be strongly connected will $\delta_t(D) \geq 4k - 1$ suffice to ensure a cyclic triangle factor?

Triangle factors - conjectures

Note that if $\delta_t(D) \geq 2\delta_0(D)$ and that if $\delta_0(D) \geq |D|/2$ then D is strongly connected.

Conjecture (Czygrinow, Kierstead, M. 2012)

If D is a digraph on $3k$ vertices and $\delta_0(D) \geq 2k$ then D contains k independent cyclic triangles.

Theorem (Corrádi-Hajnal)

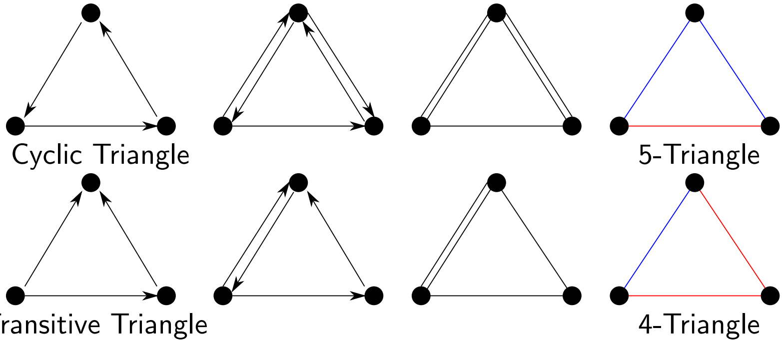
If G is a graph on $3k$ vertices and $\delta(G) \geq 2k$ then G contains k independent triangles.

Theorem (Czygrinow, Kierstead, M. 2012)

For any $\varepsilon > 0$ there exists n_0 such that the following holds. If D is a digraph on $3k \geq n_0$ vertices and $\delta_0(D) \geq (2+\varepsilon)k$ then D contains k independent cyclic triangles.

Transform to multigraph problem

For any be a digraph we can remove the orientation from to the underlying multigraph M.



Triangle factors - multigraph version

- Let $\mu_M(x, y)$ be the number of edges between x and y in M.
- Call a mulitgraph standard if $\mu_M(x, y) \leq 2$ for every vertex pair x, y.

Theorem (Triangle factor)

If D is a digraph on $3k$ vertices and $\delta_t(D) \geq 4k - 1$ then D has a perfect tiling consisting of t transitive triangles and c cyclic triangles where $t + c = k$ and $t \geq 1$

Theorem (Triangle factor (multigraph version))

If M is a standard multigraph on $3k$ vertices and $\delta(M) \geq 4k - 1$ then M has a perfect tiling consisting of $k - 1$ independent 5-triangles and one 4-triangle.

Triangle factors - simplified

Theorem (Triangle factor)

If D is a digraph on $3k$ vertices and $\delta_t(D) \geq 4k-1$ then D has a perfect tiling consisting of t transitive triangles and c cyclic triangles where $t+c=k$ and $t \geq 1$

Corollary (Transitive triangle factor)

If D is a digraph on $3k$ vertices and $\delta_t(D) \geq 4k-1$ then D has a k-independent transitive triangles.

Triangle factors - extension

Theorem (Corrádi-Hajnal)

If G is a graph on $3k$ vertices and $\delta(G) \geq 2k$ then G contains k independent triangles.

Corollary (Transitive triangle factor)

If D is a digraph on $3k$ vertices and $\delta_t(D) \geq 4k - 1$ then M has k independent transitive triangles.

Theorem (Hajnal-Szemerédi)

If G is a graph on sk vertices and $\delta(G) \geq (s-1)k$ then G contains k independent K_s.

Hajnal-Szemerédi for directed graphs

Definition (Transitive s-tournament)

An oriented of K_s that contains no directed cycles

Conjecture (Czygrinow, Kierstead, M. (trans. tournaments))

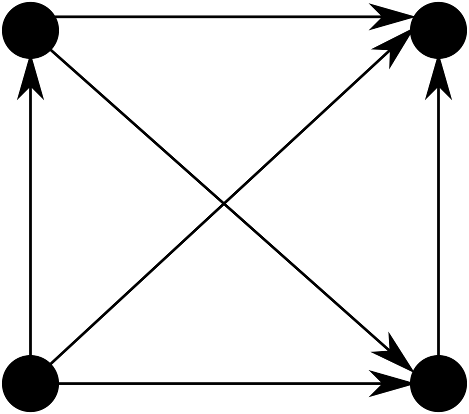
If D is a digraph on sk vertices and $\delta_t(D) \geq 2(s-1)k - 1$ then D contains k independent transitive s-tournaments.

Theorem (Czygrinow, Kierstead, M. 2012)

For any $s \geq 1$ there exists k_0 such that the following holds. If D is a directed graph on $sk \geq sk_0$ vertices and $\delta_t(D) \geq 2(s-1)k - 1$ then then D contains k independent transitive s-tournaments.

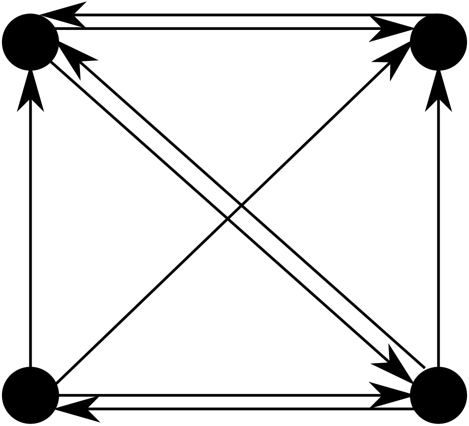
Transform to a multigraph problem

A transitive 4-tournament



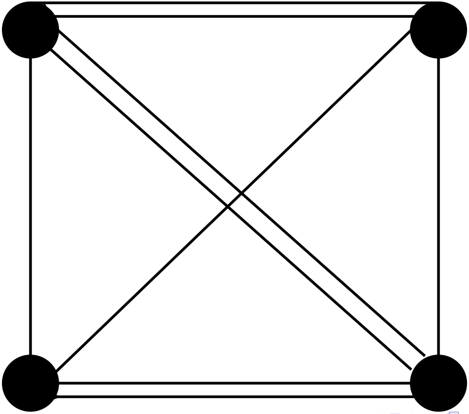
Transform to a multigraph problem

A digraph that contains a transitive 4-tournament



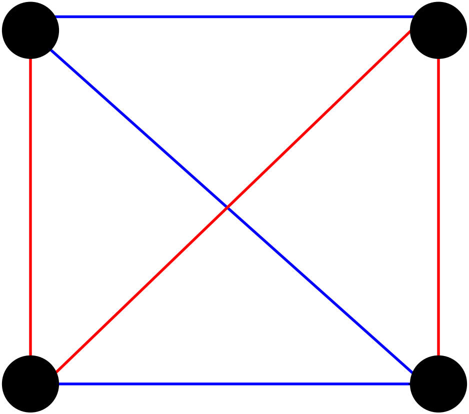
Transform to a multigraph problem

Remove the orientation - note that *any* orientation of the edges of this multigraph that gives a digraph contains a transitive 4-tournament



Transform to a multigraph problem

Color the “light” / “single” edges red and the “heavy” / “double” edges blue



M, L, H and G

Definition

- M is the multigraph obtained by removing the orientation from the edges of D. Note that M is a standard multigraph.
- $L := (V, E_L)$ where $E_L := \{e \in E(M) : \mu(e) = 1\}$
- $H := (V, E_H)$ where $E_H := \{e \in E(M) : \mu(e) = 2\}$
- $G := (V, E_L \cup E_H)$

Acyclic s-cliques.

Let K be an s-subset of $V(D)$.

Definition (acyclic s-clique)

$M[K]$ is a acyclic s-clique if $G[K]$ is clique and $L[K]$ is acyclic. Sometime we will call an acyclic s-clique a s-aclique.

Proposition

Proposition: If $M[K]$ is an acyclic s-clique then $D[K]$ contains a transitive s-tournament.

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Proof.

1. Prove by induction on s and $s \leq 3$ is trivial.
2. Let v be a leaf in $L[K]$ and pick $u \in K \setminus \{v\}$ such that $w \notin N_L(v) \cap K$ for all $w \in K \setminus \{u, v\}$.
3. By induction, there is a transitive $(s-1)$-tournament T' in $D[K - v]$.
4. If $\overrightarrow{vu} \in E(D)$, form T be adding $\overrightarrow{vw}$ to T for every $w \in K \setminus \{v\}$.
5. Otherwise, form T be adding $\overleftarrow{vw}$ to T for every $w \in K \setminus \{v\}$.

□

Statement of Theorem

Theorem (Hajnal-Szemerédi)

If G is a graph on sk vertices and $\delta(G) \geq (s-1)k$ then G contains k independent K_s.

Theorem (trans. tournaments - large graphs)

For any $s \geq 1$ there exists k_0 such that the following holds. If D is a digraph on $sk \geq sk_0$ vertices and $\delta_t(G) \geq 2(s-1)k - 1$ then D contains k independent transitive s-tournaments.

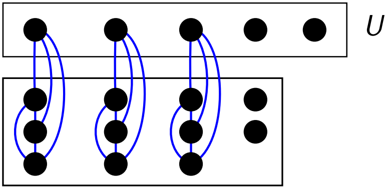
Theorem (aclique factor - large graphs - multigraphs)

For any $s \geq 1$ there exists k_0 such the following holds. If M is a standard multigraph on $sk \geq sk_0$ vertices and $\delta(M) \geq 2(s-1)k - 1$ then M contains k independent acyclic s-cliques

Lower bound example A

Example

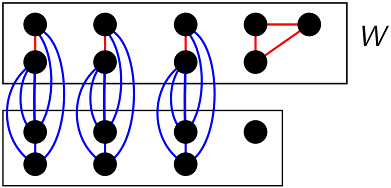
- M_A a standard multigraph on sk vertices that contains a set U of size $k+1$ such that $G[U]$ is empty and all other possible edges.
- $d(u) \geq 2(sk-(k+1)) = 2(s-1)k-2$ for any $u \in U$.
- Any acyclic s-clique in M_A can have at most one vertex in U.



Lower bound example B

Example

- M_B a standard multigraph on sk vertices that contains a set W of size $2k+1$ such that $H[W]$ is empty and all other possible edges.
- $d(w) = 2(sk-(2k+1)) + 2k = 2(s-1)k-2$ for $w \in W$.
- Each acyclic s-clique can have at most two vertices in W.



Stability method

Definition (Extremal)

Call M extremal if it is "close" to M_A or M_B.

- Handle the case when M is extremal separately.
- In the rest of the proof,that is, the non-extremal case we can assume that the graph is not "close" to M_A or M_B.
- Our proof of the non-extremal case relies on ideas in the paper "How to avoid using the Regularity Lemma: Pósa's conjecture revisited" by Levitt, Sárközy and Szemerédi

Extremal conditions

Let M be a standard multigraph on $n := sk$ vertices and let $\alpha > 0$.

Definition ((1,α)-extremal - close to M_A)

There is a set U of order k such that there are fewer than αn^2 edges in $M[U]$.

Definition ((2,α)-extremal - close to M_B)

There is a set W of order $2k$ such that there are fewer than αn^2 edges in $H[W]$.

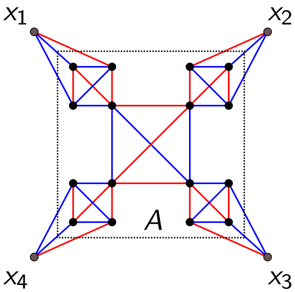
Non-extremal case - Absorbing Structure

Definition (Absorbs)

Say a vertex set A aborbs a vertex set X if there is an perfect s-aclique tiling of $M[A]$ and a perfect s-aclique tiling of $M[A \cup X]$.

Example (Absorbing $X := \{x_1, x_2, x_3, x_4\}$)

Absorbing Structure



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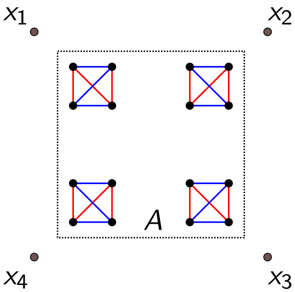
Prior to absorption



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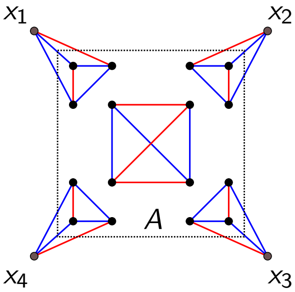
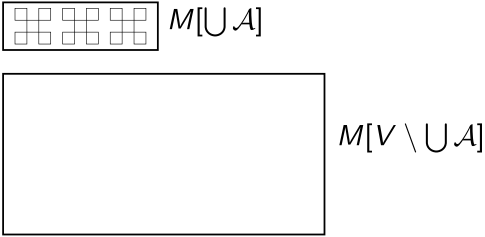
Example (Absorbing $X := \{x_1, x_2, x_3, x_4\}$)

After absorbing $\{x_1, x_2, x_3, x_4\}$.



Non-extremal - plan

1. Pick a "small" non-overlapping collection $\mathcal{A}$ of absorbing sets probabilistically so that every s-set is absorbed by a "few" elements of $\mathcal{A}$.
2. Tile almost all of $M[V \setminus \bigcup \mathcal{A}]$ with s-acliques.
3. Let $\mathcal{A}$ "absorb" the remaining vertices



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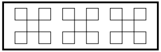
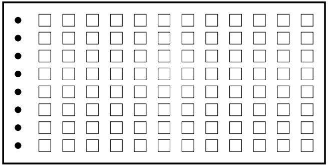


$M[\bigcup \mathcal{A}]$



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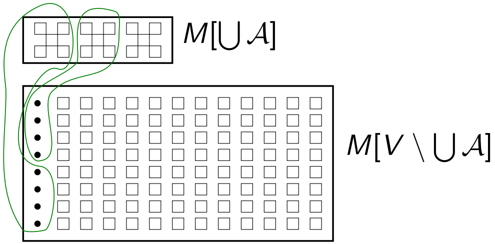
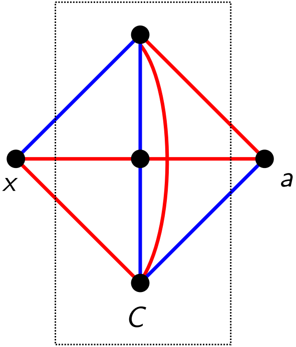


Absorbing Lemma - counting argument

- Let $M = (V, E)$ be a standard muligraph, $0 < \alpha \ll \beta \ll \gamma \ll 1$ and $n = |V|$.
- The absorbing structure has order s^2
- Need to show that for every $X \in \binom{V}{s}$ there are at least $\gamma\binom{n}{s^2}$ sets in $\binom{V}{s^2}$ that absorb X.
- Then use the Chernoff bound to show that there exists $\mathcal{A} \subseteq \binom{V}{s^2}$ such that:
  - $|\mathcal{A}| \leq \beta n$,
  - the sets in $\mathcal{A}$ do not overlap, and
  - for every s-set X, there are at least αn sets in $\mathcal{A}$ that absorb X.

Absorbing Lemma - counting argument

Between any two vertices x and a prove that there are $\Omega(\binom{n}{s-1})$ $(s-1)$-sets C such that $M[C \cup \{x\}]$ and $M[C \cup \{a\}]$ both contain acyclic s-cliques.



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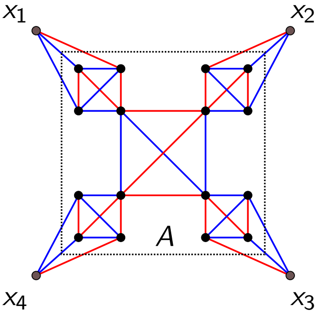

Equitable coloring

All proofs of the Hajnal-Szemerédi Theorem attack the complementary problem. That is, the equitable coloring problem.

Definition

A proper coloring is an equitable coloring if the sizes of any two color classes differ by at most 1.

Example

If G has sk vertices then in an equitable k-coloring of G each color class has size s.

Hajnal-Szemerédi - equitable coloring

- Let $\overline{G}$ be the complement of G.
- $\delta(G) + \Delta(\overline{G}) = |G| - 1$
- A clique in G corresponds to an independent set in $\overline{G}$

Theorem (Hajnal-Szemerédi - perfect tiling)

If G is a graph on sk vertices and $\delta(G) \geq (s-1)k$ then G has k independent K_s.

Theorem (Hajnal-Szemerédi - equitable coloring)

If G is graph on sk vertices and $\Delta(G) \leq k-1$ then G has an equitable k-coloring.

Acyclic cliques and equitable coloring

Let $M = (V, E)$ is a standard multigraph.

- Define $\overline{M}$ to be the standard multigraph on V in which $\mu_{\overline{M}}(xy) = 2 - \mu_M(xy)$ for every pair of vertices x, y.
- $\delta(M) + \Delta(\overline{M}) = 2|M| - 2$
- Acyclic cliques in M corresponds to a forests in $\overline{M}$.
- Call a coloring an acyclic equitable coloring if sizes of the color classes differ by at most one and the color classes induce forests.

Conjecture (perfect tiling)

If M is a standard multigraph on sk vertices and $\delta(G) \geq 2(s-1)k - 1$ then M has k independent acyclic s-cliques.

Conjecture (acyclic equitable coloring)

If M is standard multigraph on sk vertices and $\Delta(G) \leq 2k - 1$ then M has an acyclic equitable k-coloring.

Current work

- We have tried to apply Hajnal-Szemerédi type arguments to prove the case where $s = 4$.
- Will probably take a few new ideas to prove these case and the general case.