The k-on-a-path problem on claw-free graphs

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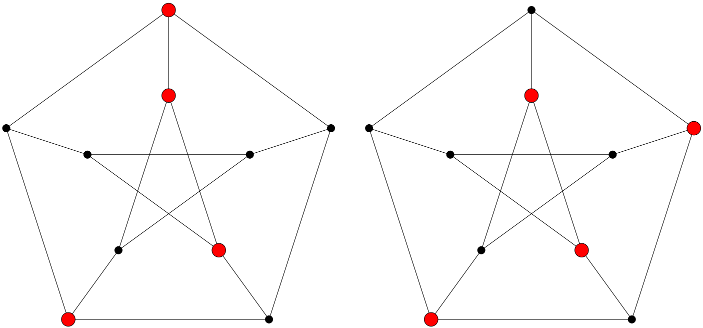
The problem: k-on-a-path

Instance: Graph G, terminals $t_1, \ldots, t_k$.
Query: Does G contain an induced path through terminals $t_1, \ldots, t_k$? (in any order)

Example:



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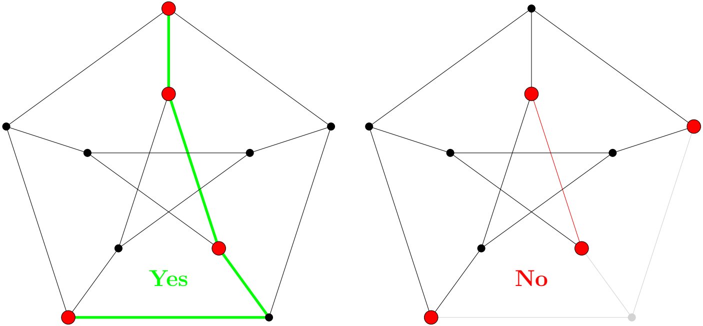
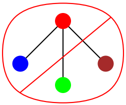
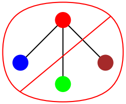


Known results on related problems

- $k = 3$, induced path, general graphs: NPc
  [Haas, Hoffmann]
- fixed k, not necessarily induced path, general graphs: P
  [Robertson, Seymour]
- Observation: Path in $G \iff$ induced path in $L(G)$

- Corollary: fixed k, induced path, line graphs: P
- $k = 4$, induced tree with 1 branching, general graphs: NPc
- $k = 3$, induced tree, general graphs: P
  [Chudnovsky, Seymour]
- Observation: on claw-free graphs
  each induced tree is an induced path

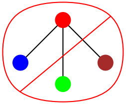


- Corollary: $k = 3$, induced path, claw-free graphs: P claw free graphs  $k = 3$

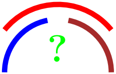
line graphs 

any fixed k

claw free graphs



proper circular-arc graphs



proper interval graphs

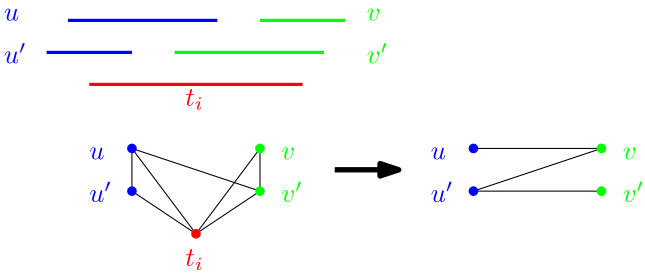
line graphs

Claim: k-on-a-path $\in$ P also on

- (proper) interval graphs
- (proper) circular-arc graphs

The order of terminals corresponds to the order of intervals from left to right or vice-versa.

To avoid induced edges between neighbors of terminals we replace neighborhoods of terminals by their complements.



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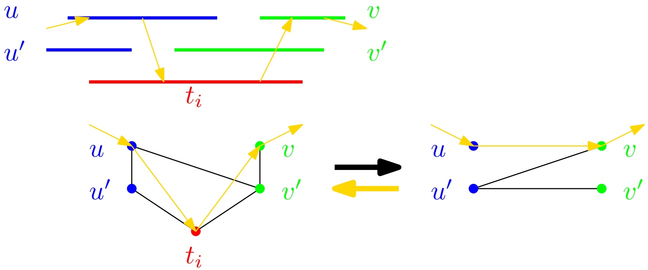


New result

> **Theorem:** The problem k-on-a-path is solvable in polynomial time on claw-free graphs for every fixed k; it becomes NP-complete, when k is a part of the input.

NP-copleteness by a reduction from the Hamiltonian path problem.

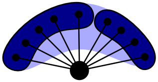
Idea of the algorithm

Simplification and reduction of the graph G:

I. reduction from claw-free to quasi-line graphs

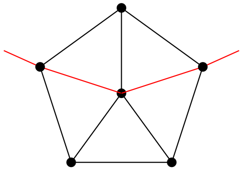
II. decomposition of quasi-line graphs

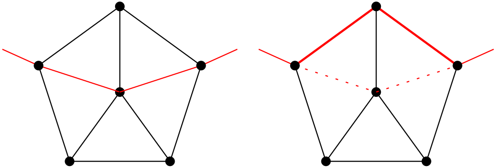
Definition: *Quasi-line graph* is a graph, where the neighborhood of each vertex can be covered by two cliques.

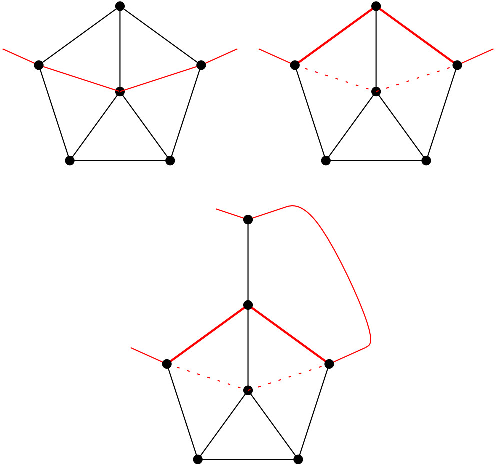


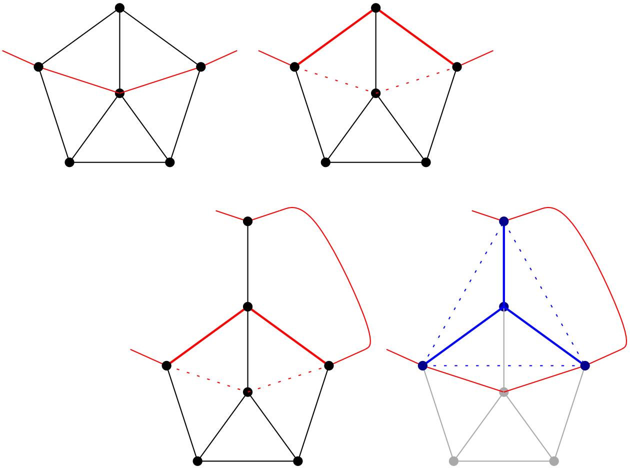
Part I: reduction to quasi-line graphs

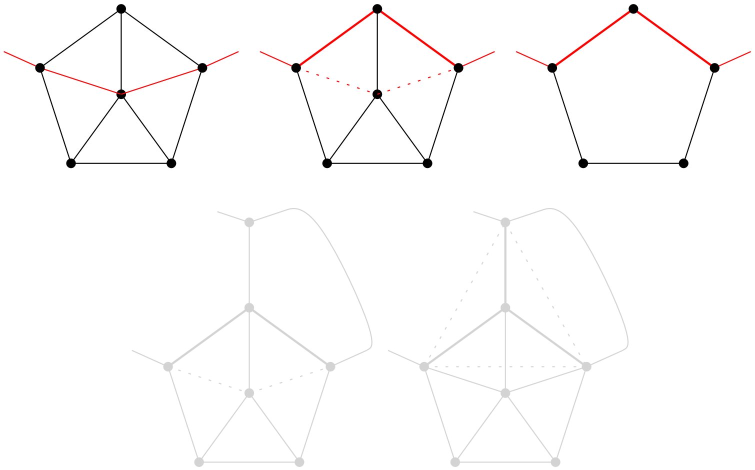
1) Restrict to a fixed order of terminals where the inner terminals are of degree 2 and the outer of degree 1.
   By brutte force we explore each of $O(k!n^{2k-2})$ possibilities.
2) Cleaning — remove vertices that do not belong to any induced path between t_1 and t_k.
   Corollary: G does not contain $\overline{C_k}$ for all odd $k \geq 7$.
   [Paulusma et al.]
3) Remove C_5 on neighborhoods — by a case analysis, how the path can be diverted.











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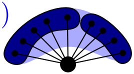
   Corollary: G does not contain $\overline{C_k}$ for odd $k \geq 7$ [Paulusma et al.].

3) Remove C_5 on neighborhoods — by a case analysis, how the path can be diverted.

G does not contain odd antiholes $\overline{C_k}$ for $k \geq 3$ on neighborhoods
$\iff \forall u \in V_G : N(u)$ is a complement of a bipartite graph
$\iff \forall u \in V_G : N(u)$ can be covered by two cliques
$\iff G$ is a quasi-line graph

$\overline{N(u)}$

$N(u)$

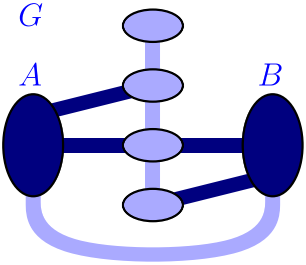


Part II: decomposition of quasi-line graphs

Theorem: A quasi-line graph with no homogeneous pair is either a proper circular arc graph or an amalgamation of interval graphs. [Chudnovsky, Seymour] Alg.: [King, Reed]

Definition: Cliques A and B form a *homogeneous pair*, if every other vertex is adjacent to whole A or not to A; resp. B.

Lemma: If G is a quasi-line graph and A, B is a homogeneous pair, then k-on-a-path can be reduced from G to G', obtained by the contraction of the cliques A and B.



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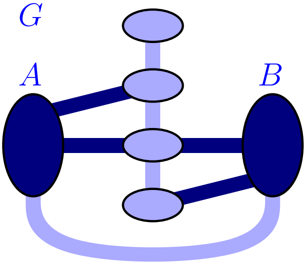

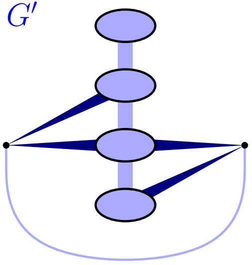

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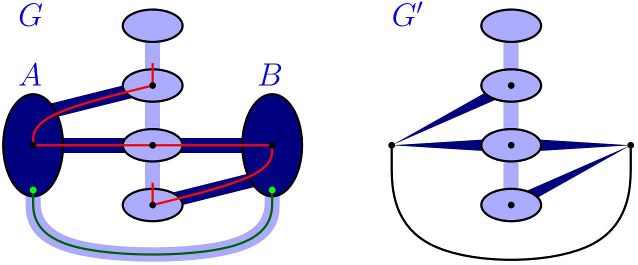

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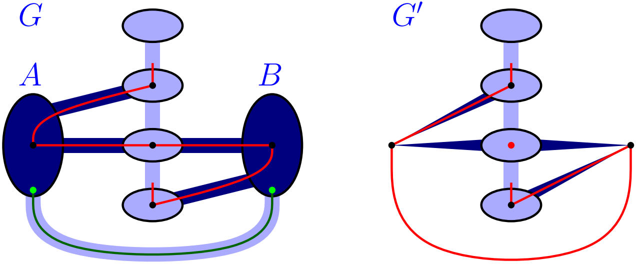
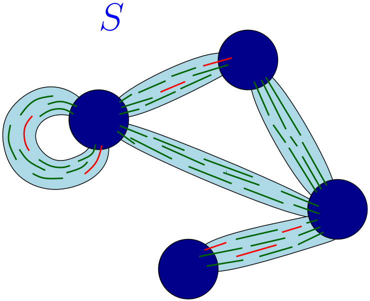


Amalgamation of interval graphs



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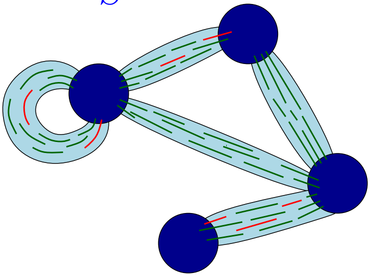
S



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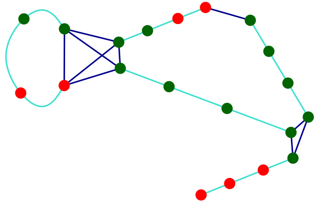

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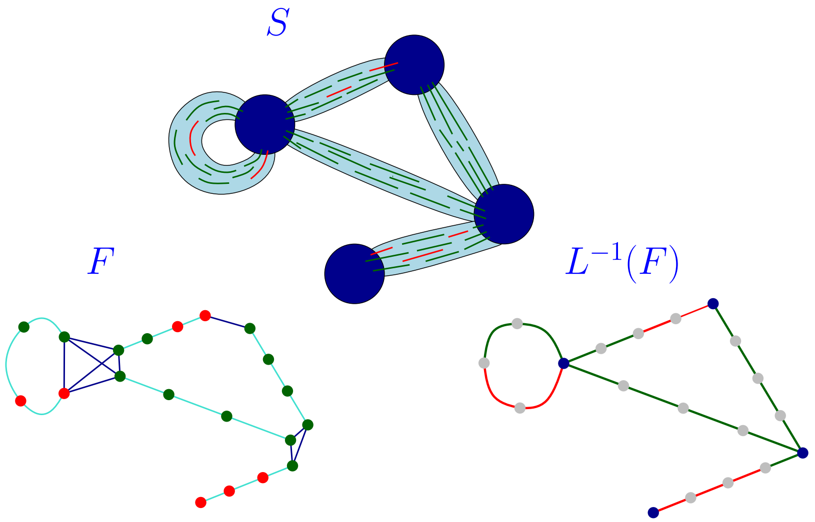
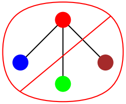
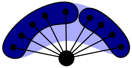
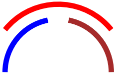
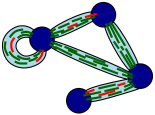


The overall story

claw free graphs 

quasi-line graphs 

proper circular-arc graphs 

amalgamations of interval graphs 

proper interval graphs 

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