

Homework #1

Spring 2019

Due Friday, January 25

1. On the first day of class, we discussed optimization problems in the standard form

$$\begin{aligned} & \underset{x_1, x_2, \dots, x_n \in \mathbb{R}}{\text{minimize}} && f(x_1, x_2, \dots, x_n) \\ & \text{subject to} && g_1(x_1, x_2, \dots, x_n) \leq 0, \\ & && \dots, \\ & && g_m(x_1, x_2, \dots, x_n) \leq 0. \end{aligned}$$

More general problems can be put in this form. Suppose you are given the *maximization* problem:

$$\begin{aligned} & \underset{x, y \in \mathbb{R}}{\text{maximize}} && x + 2y \\ & \text{subject to} && x^2 + y^2 = 1. \end{aligned}$$

Show how to write this problem in the standard form above. (I'm not asking you to solve the problem.)

2. Find the local and global minimizers of the following functions:

(a) $f(x) = x^3 + x + 1$.

(b) $f(x) = e^x - 2x$.

(c) $f(x) = \frac{x}{x^2 + 1}$.

(Hint: the quantities $x^2 - 2x + 1 = (x - 1)^2$ and $x^2 + 2x + 1 = (x + 1)^2$ cannot be negative!)

3. Compute the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 4 & -6 \\ 3 & -5 \end{bmatrix}$.

4. For two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the expression $\|t\mathbf{x} + (1 - t)\mathbf{y}\|^2$ is a quadratic function of t .

Simplify it to the form $at^2 + bt + c$, where a, b, c are in terms of \mathbf{x} and \mathbf{y} .

5. (Only 4-credit students need to do this problem.)

In this problem, we consider the function $f(x, y) = (y - x^2)(y - 2x^2)$, and its critical point at $(0, 0)$.

- (a) Let $\phi(t)$ be the restriction of $f(x, y)$ to a line in the direction (u, v) from $(0, 0)$: $\phi(t) = f(tu, tv)$.

Show that 0 is always a local minimizer of $\phi(t)$, no matter which direction (u, v) we pick.

- (b) Show that $(0, 0)$ is *not* a local minimizer of $f(x, y)$, despite it being “a local minimizer along every line through $(0, 0)$ ”.

General instructions for writing up homework:

- If you're taking the class for 4 credits (as opposed to the default of 3), write this on your assignment so that it can be graded appropriately.
- When writing up solutions, if you use a result from your textbook, say the result you're using (by name, or theorem number, or whatever) and why it applies. E.g., “So $f''(5) = 1$. By the second derivative test, since $f''(5) > 0$, the critical point $x = 5$ is a strict local minimizer.”
- You don't need to show your work for routine computations, but if you get those wrong without showing your work, you'll miss the opportunity for partial credit.
- Write proofs in complete sentences.