Math 484: Nonlinear Programming — Mikhail Lavrov

Homework #7

Fall 2018 — *Due Friday, November 2*

1. Give an example of each of the following, with justification:

   (a) Functions $f, g : \mathbb{R} \to \mathbb{R}$ such that f and g are convex, but the composition $h(x) = g(f(x))$ is not convex.

   (b) Functions $f, g : \mathbb{R} \to \mathbb{R}$ such that f is convex and g is increasing, but the composition $h(x) = g(f(x))$ is not convex.

   (c) Functions $f, g : \mathbb{R} \to \mathbb{R}$ such that f and g are convex, but the product $h(x) = f(x)g(x)$ is not convex.

   (d) A function $f : \mathbb{R}^2 \to \mathbb{R}$ of the form $f(x, y) = ax^2 + bxy + cy^2$ which is convex, but not strictly convex.

2. Use the Karush–Kuhn–Tucker theorem to solve

$$(P) \qquad \begin{cases} \underset{(x,y)\in\mathbb{R}^2}{\text{minimize}} & x \\ \text{subject to} & x^2 + y^2 - 1 \le 0, \\ & y \le 0. \end{cases}$$

3. For the convex program P in the previous problem, the value function $MP(z_1, z_2)$ has a different behavior in each of the three different regions

$$\begin{aligned} R_1 &= \{(z_1, z_2) : z_1 \ge 0 \text{ and } z_2 \ge -1\}, \\ R_2 &= \{(z_1, z_2) : z_1 \le 0 \text{ and } z_2 \ge z_1^2 - 1\}, \\ R_3 &= \{(z_1, z_2) : z_2 < \min(z_1, 0)^2 - 1\} = \mathbb{R}^2 \setminus (R_1 \cup R_2). \end{aligned}$$

   Find a formula for $MP(z_1, z_2)$ in each of the regions.

4. Explain what happens when you try to use the Karush–Kuhn–Tucker theorem to solve the convex program

$$(P) \qquad \begin{cases} \underset{(x,y)\in\mathbb{R}^2}{\text{minimize}} & x + y \\ \text{subject to} & x^2 - y + 1 \le 0, \\ & y - 2x \le 0. \end{cases}$$

5. *(Only 4-credit students need to do this problem.)*

   Prove that for any superconsistent convex program P, either $MP(\mathbf{0}) = -\infty$, or else $MP(\mathbf{z})$ is never equal to $-\infty$ for any $\mathbf{z}$.