Math 484: Nonlinear Programming | Mikhail Lavrov

Homework #5

Fall 2018 | *Due Friday, October 12*

1. Use the Gram–Schmidt process to find orthonormal vectors $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \mathbf{u}^{(3)} \in \mathbb{R}^4$ that span the same subspace of $\mathbb{R}^4$ as

$$\mathbf{a}^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{a}^{(2)} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{a}^{(3)} = \begin{bmatrix} 0 \\ 3 \\ -1 \\ 6 \end{bmatrix}.$$

2. Let $\mathbf{u}^{(1)}, \mathbf{u}^{(2)}, \dots, \mathbf{u}^{(n)}$ be an orthonormal basis of $\mathbb{R}^n$, and let $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \dots, \mathbf{v}^{(n)}$ be any vectors in $\mathbb{R}^m$. Show that the $m \times n$ matrix

$$M = \mathbf{v}^{(1)}(\mathbf{u}^{(1)})^\mathsf{T} + \mathbf{v}^{(2)}(\mathbf{u}^{(2)})^\mathsf{T} + \cdots + \mathbf{v}^{(n)}(\mathbf{u}^{(n)})^\mathsf{T}$$

is the unique matrix with the property that $M\mathbf{u}^{(i)} = \mathbf{v}^{(i)}$ for all i.

3. Find the minimum norm solution of the underdetermined linear system

$$\begin{aligned} x_1 \qquad + x_3 + x_4 &= 1, \\ 2x_1 + x_2 + x_3 + x_4 &= 0. \end{aligned}$$

4. Minimize $f(x, y, z) = 3x^2 - 6xy + 4y^2 + 6z^2$ subject to $x - y - z = 3$.

5. *(Only 4-credit students need to do this problem.)*

   We say that an $n \times n$ matrix P is a *projection matrix* if it satisfies $P^\mathsf{T} P = P$.

   (a) Given an arbitrary $m \times n$ matrix A, show that $AA^\dagger = A(A^\mathsf{T} A)^{-1} A^\mathsf{T}$ is a projection matrix by this definition.

   (b) Given an arbitrary projection matrix P, show that for any vector $\mathbf{y} \in \mathbb{R}^n$, $P\mathbf{y} - \mathbf{y}$ is orthogonal to any element of the subspace $\{P\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}$.