

## Lecture 6: The Two-Phase Simplex Method

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## 1 The problem we want to solve

In the previous lecture, we took a shortcut. When our feasible region has the form

$$\{\mathbf{Ax} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$$

and  $\mathbf{b} \geq \mathbf{0}$ , the slack variables we add to put the program in equational form form our starting basis. In other words, when all our variables are required to be nonnegative, and the point  $\mathbf{x} = \mathbf{0}$  is feasible, it is a basic feasible solution.

Suppose we have a problem such as this one:

$$\begin{aligned} & \underset{\mathbf{x} \in \mathbb{R}^4}{\text{minimize}} && x_1 + x_2 - x_3 - x_4 \\ & \text{subject to} && -3x_1 + 2x_2 + x_3 + x_4 = 7 \\ & && 2x_1 - x_2 - x_3 - 3x_4 = -1 \\ & && x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

We can definitely solve this problem with the simplex method. . . provided we have a starting basic feasible solution. But we don't know how to find one!

We could try guessing: try to see if the basic solution with  $x_1$  and  $x_2$  as the basic variables is feasible. If not, try  $x_1$  and  $x_3$ , and so on. But this reduces to the naive approach to linear programming, where we try all possible bases. (We don't even know, necessarily, that this linear program has any feasible solutions.)

The solution is the two-phase simplex method. In this method, we:

1. Solve an auxiliary problem, which has a built-in starting point, to determine if the original linear program is feasible. If we succeed, we find a basic feasible solution to the original LP.
2. From that basic feasible solution, solve the linear program the way we've done it before.

## 2 Writing down the auxiliary problem

Let's assume we're given a program in equational form: the constraints are  $\mathbf{Ax} = \mathbf{b}$  and  $\mathbf{x} \geq \mathbf{0}$ .

We begin with a minor change to the linear program whose purpose will become clear in a bit. For every constraint  $A_i \mathbf{x} = b_i$  with  $b_i < 0$ , we multiply both sides by  $-1$  to make  $b_i$  positive rather

<sup>1</sup>This document comes from the Math 482 course webpage: <https://faculty.math.illinois.edu/~mlavrov/courses/482-spring-2020.html>

than negative. In our example, we multiply the second constraint by  $-1$  and get:

$$\begin{aligned} & \underset{\mathbf{x} \in \mathbb{R}^4}{\text{minimize}} && x_1 + x_2 - x_3 - x_4 \\ & \text{subject to} && -3x_1 + 2x_2 + x_3 + x_4 = 7 \\ & && -2x_1 + x_2 + x_3 + 3x_4 = 1 \\ & && x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

This is always valid when we're dealing with equations. (It's only with inequalities that multiplying by  $-1$  could mess things up.)

Next, we add new nonnegative variables  $x_1^a, x_2^a, \dots$  called *artificial variables*, one for every constraint. We modify the existing constraints by adding  $x_i^a$  to the left-hand side of the  $i^{\text{th}}$  constraint. So our constraints become:

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_4 + x_1^a &= 7 \\ -2x_1 + x_2 + x_3 + 3x_4 + x_2^a &= 1 \\ x_1, x_2, x_3, x_4, x_1^a, x_2^a &\geq 0. \end{aligned}$$

In general,  $A\mathbf{x} = \mathbf{b}$  becomes  $A\mathbf{x} + I\mathbf{x}^a = \mathbf{b}$ , where  $\mathbf{x}^a$  is the vector of artificial variables.

So, on the one hand, there are plenty of solutions to this new system that have nothing to do with the old one. On the other hand, there's no question of feasibility. If we set  $\mathbf{x} = \mathbf{0}$  and  $\mathbf{x}^a = \mathbf{b}$ , then we get a feasible solution! (In the example, this means setting  $x_1 = x_2 = x_3 = x_4 = 0$ ,  $x_1^a = 7$ , and  $x_2^a = 1$ .) This is why we wanted to make sure that  $\mathbf{b} \geq \mathbf{0}$ : this way, the nonnegativity constraints on  $\mathbf{x}^a$  are satisfied when we do this.

Moreover, even though we've departed from the original problem a little, the new problem is not completely unrelated. When  $x_1^a = x_2^a = 0$  (and in general, when  $\mathbf{x}^a = \mathbf{0}$ ), the new constraints reduce to the old constraints. This is the ideal state we'd like to be in: if we can get here, we've found a feasible solution of the original problem.

We can try to see if it's even possible to get to this state by minimizing the artificial objective function  $z^a = x_1^a + x_2^a + \dots + x_m^a$ : the sum of all the artificial variables. In the example, this means solving the linear program

$$\begin{aligned} & \underset{\mathbf{x} \in \mathbb{R}^4, \mathbf{x}^a \in \mathbb{R}^2}{\text{minimize}} && x_1^a + x_2^a \\ & \text{subject to} && -3x_1 + 2x_2 + x_3 + x_4 + x_1^a = 7 \\ & && -2x_1 + x_2 + x_3 + 3x_4 + x_2^a = 1 \\ & && x_1, x_2, x_3, x_4, x_1^a, x_2^a \geq 0. \end{aligned}$$

The smallest conceivable value of  $x_1^a + x_2^a$  in the optimal solution to this linear program is 0, and that can only be achieved if  $x_1^a = x_2^a = 0$ . If we find an optimal solution to this linear program and its objective value is 0, great! In that case, we can take that optimal solution, throw away the artificial variables, and get a feasible solution to the original linear program.

What if the smallest possible value of  $x_1^a + x_2^a$  turns out to be something positive? Well, in that case, we know that there can't be a feasible solution to the original linear program: any such solution would give a value of 0 in the new linear program. So in this case, you complain to whoever gave you that linear program, and tell them that you can't satisfy all of their constraints.

### 3 Setting up the tableau and solving

Recording all of this information in a tableau, we do things slightly differently. The first two rows are just the usual recording of the constraints; we make  $x_i^a$  the basic variable of the  $i^{\text{th}}$  constraint. However, we include rows for *both* objective functions:  $z$ , the original objective function, and  $z^a$ , the artificial objective function. We do this because eventually we'll want to switch back to the original linear program, and when we do, we'll want to keep  $z$  around.

$$\begin{array}{cccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_1^a & x_2^a & \\
 \hline
 x_1^a & -3 & 2 & 1 & 1 & 1 & 0 & 7 \\
 x_2^a & -2 & 1 & 1 & 3 & 0 & 1 & 1 \\
 \hline
 -z & 1 & 1 & -1 & -1 & 0 & 0 & 0 \\
 -z^a & 0 & 0 & 0 & 0 & 1 & 1 & 0
 \end{array}$$

Our first step here is always to finish the job of row-reducing. Everything in the artificial variables' columns is cleaned up, except for the  $-z^a$  row. Subtract the  $x_1^a$  and  $x_2^a$  rows from that row, and we get:

$$\begin{array}{cccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_1^a & x_2^a & \\
 \hline
 x_1^a & -3 & 2 & 1 & 1 & 1 & 0 & 7 \\
 x_2^a & -2 & 1 & 1 & 3 & 0 & 1 & 1 \\
 \hline
 -z & 1 & 1 & -1 & -1 & 0 & 0 & 0 \\
 -z^a & 5 & -3 & -2 & -4 & 0 & 0 & -8
 \end{array}$$

We are minimizing  $z^a$ , so we want to bring in a variable with a negative entry in the  $-z^a$  row. Let's use  $x_2$ . Both  $x_1^a$  and  $x_2^a$  have positive entries in its column, and  $x_2^a$  has the smaller ratio of  $\frac{1}{1}$ , so  $x_2$  replaces  $x_2^a$  as the new basic variable for the second row. We row-reduce:

$$\begin{array}{cccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_1^a & x_2^a & \\
 \hline
 x_1^a & -3 & 2 & 1 & 1 & 1 & 0 & 7 \\
 & -2 & \mathbf{1} & 1 & 3 & 0 & 1 & 1 \\
 \hline
 -z & 1 & 1 & -1 & -1 & 0 & 0 & 0 \\
 -z^a & 1 & -3 & -2 & -4 & 0 & 0 & -8
 \end{array}
 \rightsquigarrow
 \begin{array}{cccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_1^a & x_2^a & \\
 \hline
 x_1^a & 1 & 0 & -1 & -5 & 1 & -2 & 5 \\
 x_2 & -2 & \mathbf{1} & 1 & 3 & 0 & 1 & 1 \\
 \hline
 -z & 3 & 0 & -2 & -4 & 0 & -1 & -1 \\
 -z^a & -1 & 0 & 1 & 5 & 0 & 3 & -5
 \end{array}$$

Now  $x_1$  has a negative entry in the  $-z^a$  row, so we pivot to bring that in. In  $x_1$ 's column,  $x_1^a$  has the only positive entry, so  $x_1^a$  is the only possible leaving variable. Pivoting, we get:

$$\begin{array}{cccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_1^a & x_2^a & \\
 \hline
 & \mathbf{1} & 0 & -1 & -5 & 1 & -2 & 5 \\
 x_2 & -2 & 1 & 1 & 3 & 0 & 1 & 1 \\
 \hline
 -z & 3 & 0 & -2 & -4 & 0 & -1 & -1 \\
 -z^a & -1 & 0 & 1 & 5 & 0 & 3 & -5
 \end{array}
 \rightsquigarrow
 \begin{array}{cccccc|c}
 & x_1 & x_2 & x_3 & x_4 & x_1^a & x_2^a & \\
 \hline
 x_1 & \mathbf{1} & 0 & -1 & -5 & 1 & -2 & 5 \\
 x_2 & 0 & 1 & -1 & -7 & 2 & -3 & 11 \\
 \hline
 -z & 0 & 0 & 1 & 16 & -3 & 5 & -16 \\
 -z^a & 0 & 0 & 0 & 0 & 1 & 1 & 0
 \end{array}$$

Now we have a basic feasible solution  $(x_1, x_2, x_3, x_4) = (1, 3, 0, 0)$  to the original linear program! The artificial variables have all successfully been driven out.

To celebrate, let's rewrite the tableau, but erase all mention of anything artificial. This means:

- Deleting the  $x_1^a$  and  $x_2^a$  columns.
- Deleting the  $-z^a$  row.

This gives us

	$x_1$	$x_2$	$x_3$	$x_4$	
$x_1$	1	0	-1	-5	5
$x_2$	0	1	-1	-7	11
$-z$	0	0	1	16	-16

Now we are left with an ordinary linear program to solve. We'd proceed as usual, but coincidentally, the second phase is easy. We are minimizing  $z$ , and all the reduced costs are positive, so we're already at the optimal solution.

We conclude that the current basic feasible solution  $(x_1, x_2, x_3, x_4) = (5, 11, 0, 0)$  achieves the minimum possible objective value of  $z = 16$ .

## 4 Complications

What happens when a linear program is infeasible? In this case, we will not be able to set all artificial variables to 0. We detect infeasibility by seeing that in the optimal solution to the stage one linear program, the artificial objective  $z^a$  is still positive.

There are several other complications that can occur.

Sometimes, when minimizing  $z^a$ , we may end with a basic solution where  $z^a = 0$ , and therefore  $\mathbf{x}^a = \mathbf{0}$ , and yet have some of the artificial variables still be basic. To witness this happen, imagine that our first constraint had a RHS of 2 instead of 7; then after the first pivot, it would have a RHS of 0 instead of 5, and we'd get the tableau below (on the left):

	$x_1$	$x_2$	$x_3$	$x_4$	$x_1^a$	$x_2^a$		
$x_1^a$	1	0	-1	-5	1	-2	0	
$x_2$	-2	1	1	3	0	1	1	$\rightsquigarrow$
$-z$	3	0	-2	-4	0	-1	-1	
$-z^a$	-1	0	1	5	0	3	0	

	$x_1$	$x_2$	$x_3$	$x_4$	
	1	0	-1	-5	0
$x_2$	-2	1	1	3	1
$-z$	3	0	-2	-4	-1

Here, we've obtained a basic feasible solution  $(x_1, x_2, x_3, x_4) = (0, 1, 0, 0)$ , but the tableau is missing a basic variable in the first row.

This can be fixed very easily. Just pick any of the variables with a nonzero entry in that row, and divide through by that entry to make that the basic variable. Then row-reduce. In this case, we could pick any of  $x_1$ ,  $x_3$ , or  $x_4$ ; it's easiest to pick  $x_1$  (which has an entry of 1 there already) or  $x_3$  (which has an entry of  $-1$ , and dividing by  $-1$  is easy). We don't have to worry about dividing through by negative number if we have to; the RHS is 0, which won't become negative no matter how we scale this row.

Finally, in rare cases, this might be impossible, because the row with no basic variable is filled with 0 entries only. If this happens, it means we've discovered that one of the equations is redundant. Just delete that row (since it contains no useful information) and proceed.

## 5 Optional: a shortcut when dealing with slack variables

(We probably won't have time in class to talk about this topic, and it's not necessary to understand the two-phase simplex method, but in some cases it can save us a bit of work.)

Suppose that we're using the two-phase simplex method on a problem which started out in inequality form, where we added slack variables:

$$\left\{ \begin{array}{l} x + y \leq 3 \\ x - 2y \leq 1 \\ -x - y \leq -1 \\ x, y \geq 0 \end{array} \right. \rightsquigarrow \left\{ \begin{array}{l} x + y + s_1 = 3 \\ x - 2y + s_2 = 1 \\ -x - y + s_3 = -1 \\ x, y, s_1, s_2, s_3 \geq 0. \end{array} \right.$$

Our algorithm would tell us to multiply the last row by  $-1$  (which is fine) and then to add artificial variables to each row (which seems wasteful). Instead, how about we only add an artificial variable  $x_3^a$  to the last row, and use the slack variables as the basic variables?

Now we have a stage-one problem

$$\begin{array}{ll} \underset{x, y, s_1, s_2, s_3, x_3^a}{\text{minimize}} & x_3^a \\ \text{subject to} & x + y + s_1 = 3 \\ & x - 2y + s_2 = 1 \\ & x + y - s_3 + x_3^a = 1 \\ & x, y, s_1, s_2, s_3, x_3^a \geq 0 \end{array}$$

which can start with the basic feasible solution  $(x, y, s_1, s_2, s_3, x_3^a) = (0, 0, 3, 1, 0, 1)$ . Only one artificial variable needs to be eliminated before we can start on the original problem.