

Lecture 5: An Example of the Simplex Method

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1 An example with nothing weird going on

Consider the linear program

$$\begin{aligned}
 & \underset{x,y \in \mathbb{R}}{\text{maximize}} && 2x + 3y \\
 & \text{subject to} && -x + y \leq 3 \\
 & && x - 2y \leq 2 \\
 & && x + y \leq 7 \\
 & && x, y \geq 0.
 \end{aligned}$$

This is not in equational form, but we can easily put it in equational form by adding slack variables:

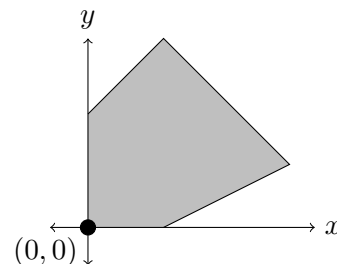
$$\begin{aligned}
 & \underset{x,y \in \mathbb{R}}{\text{maximize}} && 2x + 3y \\
 & \text{subject to} && -x + y + s_1 = 3 \\
 & && x - 2y + s_2 = 2 \\
 & && x + y + s_3 = 7 \\
 & && x, y, s_1, s_2, s_3 \geq 0.
 \end{aligned}$$

Adding slack variables has a convenient bonus effect. The slack variables (s_1, s_2, s_3) form a convenient set of basic variables to start with, for two reasons:

- The tableau will already be row-reduced for the slack variables, since each one shows up in only one equation. This will be true any time we add slack variables.
- The basic solution is $(x, y, s_1, s_2, s_3) = (0, 0, 3, 2, 7)$, which is feasible. This happens whenever we start with inequalities $A\mathbf{x} \leq \mathbf{b}$, where $\mathbf{b} \geq \mathbf{0}$. So it's not always useful, but sometimes makes our lives easier.

Here is the resulting tableau, and a graph of the feasible region (of the original LP in x and y) with the corresponding basic feasible solution marked:

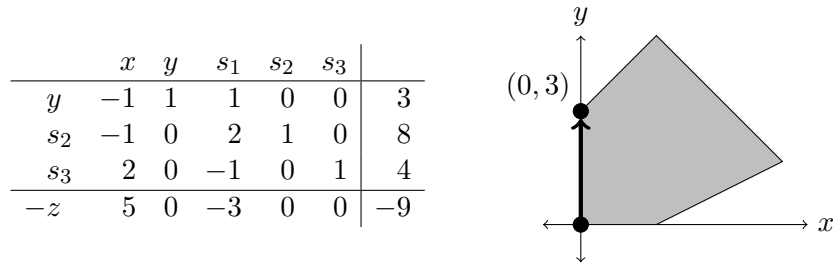
	x	y	s_1	s_2	s_3	
s_1	-1	1	1	0	0	3
s_2	1	-2	0	1	0	2
s_3	1	1	0	0	1	7
$-z$	2	3	0	0	0	0



¹This document comes from the Math 482 course webpage: <https://faculty.math.illinois.edu/~mlavrov/courses/482-spring-2020.html>

In a well-behaved 2-dimensional linear program, exactly 2 constraints should be tight at each corner. Specifically, it's the ones corresponding to the nonbasic variables. In this tableau, the nonbasic variables are x and y , and the constraints they "own" are the $x \geq 0$ and $y \geq 0$ constraints. So we should be at the corner where these two constraints are tight: the intersection of $x = 0$ and $y = 0$.

Let's pivot on y to bring it into the basis. (This is an arbitrary choice: we could also have pivoted on x .) Since s_2 's coefficient of y is -2 , it's not a valid leaving variable; s_1 and s_3 have ratios of $\frac{3}{1}$ and $\frac{7}{1}$, of which the smallest is 3. So y replaces s_1 in the basis, giving us the new tableau (and new corner point) below:

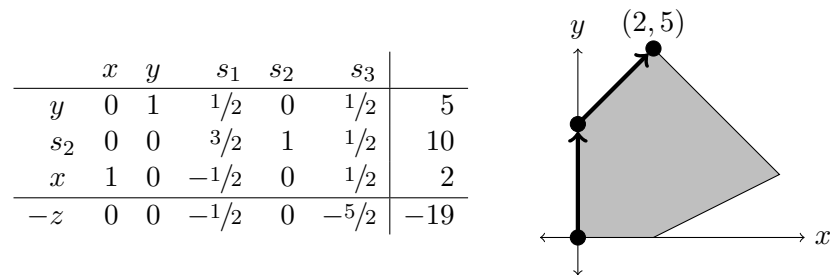


The basic feasible solution is $(x, y, s_1, s_2, s_3) = (0, 3, 0, 8, 4)$. The nonbasic variables of the new tableau are x and s_1 . As before, x "owns" the $x \geq 0$ constraint. Meanwhile, s_1 "owns" the $s_1 \geq 0$ constraint, but in the original linear program, this was the $-x + y \leq 3$ constraint. So we should be at the corner where $x = 0$ and $-x + y = 3$ meet, and indeed, these lines intersect at $(0, 3)$.

The choice of pivot variable corresponded to picking the direction in which we went around the polygon: which edge out of $(0,0)$ we used. The edge from $(0,0)$ to $(0,3)$ moves away from the $y \geq 0$ constraint, so y is the variable that becomes basic. We could also have brought x into the basis, moving away from the $x \geq 0$ constraint.

But now, at $(0,3)$, there is only one good choice of pivot. We don't want to go back to $(0,0)$, so the only choice is to continue going clockwise. In the tableau, this corresponds to how we don't want to pivot on s_1 (its reduced cost is negative, so pivoting would decrease z). Instead, the only helpful choice of pivot is x , whose reduced cost is positive.

In x 's column, the coefficients of y and s_2 are both negative, so those can't be leaving variables. Therefore x replaces s_3 in the basis, giving us the new tableau (and new corner point) below:



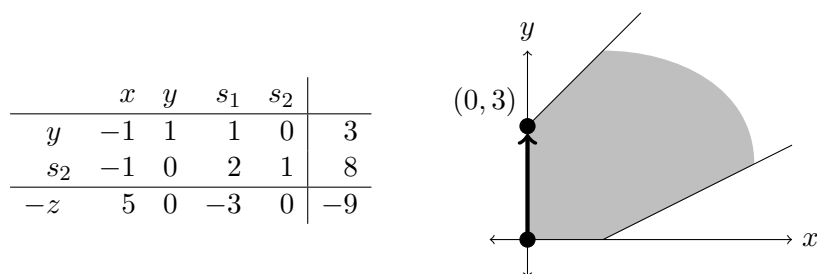
Now all reduced costs are negative, so z is maximized and $(2, 5)$ is the optimal solution.

(Here, s_1 and s_3 are nonbasic. The constraints they “own” are the $-x + y \leq 3$ constraint and the $x + y \leq 7$ constraint. So we end up at the corner point where the lines $-x + y = 3$ and $x + y = 7$ intersect.)

If we had decided to pivot on x first, we would have arrived at the same final answer, but going counterclockwise around the feasible region instead. There would have been three pivot steps, not two, because there are three edges to take when going around that way.

2 An unbounded linear program

Let’s go back a step, to when we were at the point $(0, 3)$. But now, suppose that the constraint $x + y \leq 7$ (corresponding to the variable s_3) didn’t exist. Then the tableau, and feasible region, would look like this:



The feasible region is unbounded in the direction we want to go.

It’s still a good idea to pivot on x . But now, both basic variables are ruled out at the first stage: both of them have a negative coefficient in x ’s column. There is no leaving variable to choose.

This is what it looks like when the linear program is unbounded, and z can be made arbitrarily large. There is no optimal solution.

From this tableau, we can also learn a bit about *how* the linear program is unbounded. To do this, let’s write down the equations for what happens when we increase x (the entering variable) while keeping s_1 (the other nonbasic variable) 0. Then we have

$$\begin{cases} -x + y + \cancel{s_1} = 3 \\ -x + \cancel{2s_1} + s_2 = 8 \end{cases} \implies \begin{cases} y = x + 3 \\ s_2 = x + 8 \end{cases}$$

In other words, as we pivot on x , we travel along the line $y = x + 3$; the full solution, with slack variables, varies as $(x, y, s_1, s_2) = (x, x + 3, 0, x + 8)$.

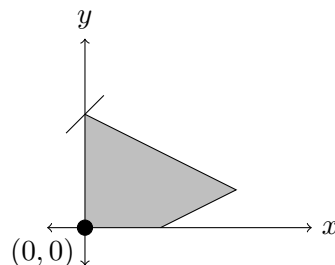
All this is happening behind the scenes when we do any pivot step of the simplex method. But here, because the coefficients in x ’s column were both negative, the slopes of $y = x + 3$ and $s_2 = x + 8$ are both positive, which means that we can increase x without a limit. And since x had a positive reduced cost of 5, we know that this gives us arbitrarily large objective values: more precisely, the point $(x, y, s_1, s_2) = (x, x + 3, 0, x + 8)$ has an objective value of $z = 5x + 9$.

Whenever we learn from the tableau that the linear program is unbounded, we can perform such an analysis to find an infinite ray of feasible solutions along which the objective value improves without bound.

3 An example of degenerate pivoting

Now suppose we replace the $x + y \leq 7$ constraint by the constraint $x + 2y \leq 6$. Here is the initial tableau and a graph of the feasible region:

	x	y	s_1	s_2	s_3	
s_1	-1	1	1	0	0	3
s_2	1	-2	0	1	0	2
s_3	1	2	0	0	1	6
$-z$	2	3	0	0	0	0

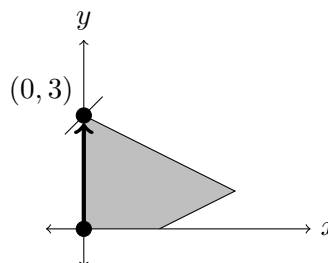


The constraint $-x + y \leq 3$ is just barely irrelevant. The line $-x + y = 3$ touches the feasible region at the corner point $(3, 0)$, and doesn't change the feasibility of anything.

We can proceed as before and bring y into the basis. As before, s_2 is out of consideration because it has a negative coefficient in y 's column. Meanwhile, s_1 and s_3 have ratios $\frac{3}{1}$ and $\frac{6}{2}$, so they are tied for having the smallest ratio. As we increase y , s_1 and s_3 will decrease and hit 0 at the same time.

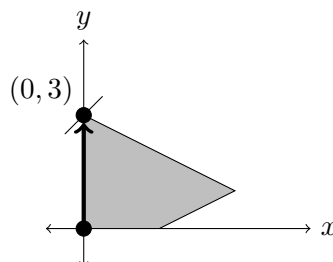
Which variable do we choose in such a case? Right now, we have no way to tell which choice is better, but either choice will give us a valid tableau. Let's choose s_1 , because that's what we did last time. We get:

	x	y	s_1	s_2	s_3	
y	-1	1	1	0	0	3
s_2	-1	0	2	1	0	8
s_3	3	0	-2	0	1	0
$-z$	5	0	-3	0	0	-9



Here is where things start to go wrong. We should pivot on x , because it's the only variable with positive reduced cost. Both y and s_2 have negative coefficients in x 's column, so the leaving variable must be s_3 . But when we make this happen, the values of the variables don't change!

	x	y	s_1	s_2	s_3	
y	0	1	$\frac{1}{3}$	0	$\frac{1}{3}$	3
s_2	0	0	$\frac{4}{3}$	1	$\frac{1}{3}$	8
x	1	0	$-\frac{2}{3}$	0	$\frac{1}{3}$	0
$-z$	0	0	$\frac{1}{3}$	0	$-\frac{5}{3}$	-9



The problem is that the three lines $x = 0$, $-x + y = 3$, and $x + 2y = 6$ all meet at the point $(0, 3)$. Previously, when x and s_1 were nonbasic, we thought of $(0, 3)$ as the intersection of $x = 0$ and

$-x + y = 3$. Now, s_1 and s_3 are nonbasic, and we've "moved" to the intersection of $-x + y = 3$ and $x + 2y = 6$. This is also $(0, 3)$.

This is called *degenerate pivoting*. There's one big problem with degenerate pivoting:

- Usually, we can say: the simplex method is always improving the value of z , so it can never revisit a vertex, and since there's only finitely many vertices, it *has* to reach the right one eventually.
- With degenerate pivoting, the value of z does not always improve. So we have no guarantee that the simplex method won't keep going forever, stuck at the same vertex being represented in different ways.

In this example, we'll leave the point $(0, 3)$ after one more step. But in more complicated examples, when many constraints meet at one vertex, staying at that vertex forever is a real danger. To avoid this, we'll need to develop pivoting rules that avoid infinite loops, by telling us the right variable to remove from the basis in cases when there's a tie.