

Lecture 34: The Cutting Plane Method

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1 Cutting planes in general

The cutting plane method is really a family of strategies for solving integer programs.

The philosophy is that an integer program can have multiple linear programming formulations: different sets of linear inequalities describing the same set of integer points. When we solve the linear programming relaxation and get a fractional solution, that's a sign that our linear programming formulation wasn't very good.

So why not try to improve it?

Specifically, suppose that we have an integer program

$$\begin{aligned} & \underset{\mathbf{x} \in \mathbb{Z}^n}{\text{maximize}} && \mathbf{c}^\top \mathbf{x} \\ & \text{subject to} && \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & && \mathbf{x} \geq \mathbf{0} \end{aligned}$$

We solve the LP relaxation, and get a fractional solution \mathbf{x}^* . If we want an improved formulation of this integer program, we want to generate a new inequality $\boldsymbol{\alpha}^\top \mathbf{x} \leq \beta$ such that:

- It's *valid* for the integer program: every point $\mathbf{x} \in \mathbb{Z}^n$ that satisfies $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ also satisfies the new inequality.

We don't want to change the problem, after all!

- It cuts off the fractional solution \mathbf{x}^* we got previously: we have $\boldsymbol{\alpha}^\top \mathbf{x}^* > \beta$.

If this does not hold, then adding the new inequality won't help; we'll still get \mathbf{x}^* as the optimal solution to the LP relaxation.

Such an inequality is called a *cutting plane* for the integer program. If we can come up with a cutting plane, then we can add it as an additional constraint, and solve the LP relaxation of the new integer program.

Of course, there's no guarantee that the new LP relaxation will have an integer solution, either. We might get another fractional solution, in which case we'll have to do this again. The hope is that after several steps, we'll get an integer solution.

One big question remains: where do we actually get these cutting planes to begin with?

There are many strategies, and each one results in a cutting plane method. They vary in how difficult they are (some require more or less work to come up with a cut) and how effective.

¹This document comes from the Math 482 course webpage: <https://faculty.math.illinois.edu/~mlavrov/courses/482-spring-2020.html>

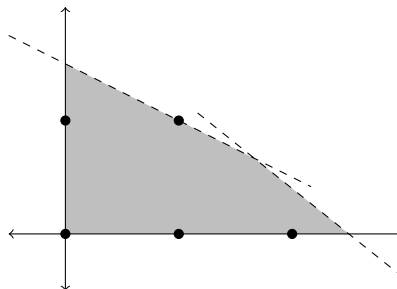
2 The Gomory fractional cut

The Gomory fractional cut is one strategy for coming up with cutting planes. It's quick to perform, and the cuts it produces aren't too bad.

To make it work, we'll assume that we have a purely integer program: some cutting plane methods still work with a mix of integer and real variables, but this one isn't one of them. We'll also assume that all coefficients in the constraints are integers. (If they're merely rational numbers, we can turn them into integers by clearing denominators.)

As an example, consider the following integer program:

$$\begin{aligned} & \underset{x,y \in \mathbb{Z}}{\text{maximize}} && 2x + 3y \\ & \text{subject to} && x + 2y \leq 3 \\ & && 4x + 5y \leq 10 \\ & && x, y \geq 0 \end{aligned}$$



A key fact about this integer program, and the reason we require all coefficients in the constraints to be integers, is that when we convert it to the equational form

$$\begin{cases} x + 2y + s_1 = 3 \\ 4x + 5y + s_2 = 10 \end{cases}$$

the slack variables s_1 and s_2 are also integers.

As before, we begin by solving the linear programming relaxation:

$$\begin{array}{c|cccc|c} x & y & s_1 & s_2 & & \\ \hline s_1 & 1 & 2 & 1 & 0 & 3 \\ s_2 & 4 & 5 & 0 & 1 & 10 \\ -z & 2 & 3 & 0 & 0 & 0 \end{array} \rightsquigarrow \begin{array}{c|cccc|c} x & y & s_1 & s_2 & & \\ \hline y & 1/2 & 1 & 1/2 & 0 & 3/2 \\ s_2 & 3/2 & 0 & -5/2 & 1 & 5/2 \\ -z & 1/2 & 0 & -3/2 & 0 & -9/2 \end{array} \rightsquigarrow \begin{array}{c|cccc|c} x & y & s_1 & s_2 & & \\ \hline y & 0 & 1 & 4/3 & -1/3 & 2/3 \\ x & 1 & 0 & -5/3 & 2/3 & 5/3 \\ -z & 0 & 0 & -2/3 & -1/3 & -16/3 \end{array}$$

If we did not find an integer solution, then by definition, one of the rows of the tableau will have a fraction in the rightmost column. In our case, both rows do; it doesn't matter which one we pick, so let's pick the first one. We can write it out as an equation:

$$y + \frac{4}{3}s_1 - \frac{1}{3}s_2 = \frac{2}{3}.$$

To deduce the fractional cut, we first split the left-hand side of this equation into two parts: an *integer* part, and a *nonnegative* part. In this example, that can be done as follows:

$$\underbrace{y + s_1 - s_2}_{\text{integer}} + \underbrace{\frac{1}{3}s_1 + \frac{2}{3}s_2}_{\text{nonnegative}} = \frac{2}{3}.$$

The integer part is an integer because we combine integer variables with integer coefficients. The nonnegative part is nonnegative because we combine nonnegative variables with nonnegative coefficients. Note that to accomplish this, we split up $\frac{4}{3}s_1$ into $s_1 + \frac{1}{3}s_2$, and $-\frac{1}{3}s_2$ into $-s_2 + \frac{2}{3}s_2$.

If we drop the nonnegative part of the left-hand side, it decreases, and so we get an inequality:

$$y + s_1 - s_2 \leq \frac{2}{3}.$$

But remember that $y + s_1 - s_2$ is an integer. If an integer is at most $\frac{2}{3}$, then it is actually at most 0. So we can deduce a sharper inequality: actually,

$$y + s_1 - s_2 \leq 0.$$

This is Gomory's fractional cut: the inequality we're going to add to our problem. Note that the same inequality can be stated in several different ways:

- We've said that the integer part of the equation is at most the integer part of the right-hand side. Equivalently, the nonnegative part of the equation is at least the fractional part of the right-hand side:

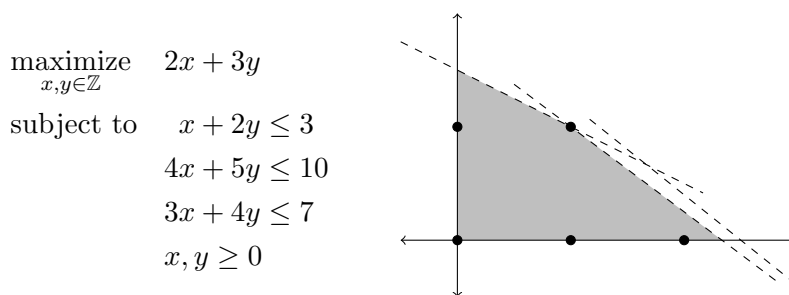
$$\frac{1}{3}s_1 + \frac{2}{3}s_2 \geq \frac{2}{3}.$$

This version is the one that people normally use, actually; it's more convenient, because when we add it to the tableau, we won't have to row-reduce to clean up the pivot columns.

- If we want to see the effect of the cut on the feasible region, it's nice to express it in terms of x and y . We can express s_1 and s_2 in terms of x and y as $s_1 = 3 - x - 2y$ and $s_2 = 10 - 4x - 5y$. Therefore $y + s_1 - s_2 \leq 0$ is equivalent to

$$y + (3 - x - 2y) - (10 - 4x - 5y) \leq 0 \implies 3x + 4y \leq 7.$$

If we wanted to draw a pretty picture, we'd use the last of these inequalities. We'd get



The first form of the cut is also useful: if we put it in equational form as $y + s_1 - s_2 + s_3 = 0$, it's easy to see that the slack variable s_3 is also an integer.

But the second form of the cut is the one we add to the tableau, after rewriting it as

$$\frac{1}{3}s_1 + \frac{2}{3}s_2 \geq \frac{2}{3} \implies -\frac{1}{3}s_1 - \frac{2}{3}s_2 \leq -\frac{2}{3} \implies -\frac{1}{3}s_1 - \frac{2}{3}s_2 + s_3 = -\frac{2}{3}.$$

This is guaranteed to contain none of the basic variables in our tableau: their coefficients in every row are either 0 or 1, so the "integer" half of the left-hand side eats all of them, and they don't contribute to the "nonnegative" half.

We expand our tableau, adding the new inequality:

$$\begin{array}{cccc|c}
 x & y & s_1 & s_2 & \\
 \hline
 y & 0 & 1 & 4/3 & -1/3 & 2/3 \\
 x & 1 & 0 & -5/3 & 2/3 & 5/3 \\
 -z & 0 & 0 & -2/3 & -1/3 & -16/3
 \end{array}
 \rightsquigarrow
 \begin{array}{cccc|ccc}
 x & y & s_1 & s_2 & s_3 & \\
 \hline
 y & 0 & 1 & 4/3 & -1/3 & 0 & 2/3 \\
 x & 1 & 0 & -5/3 & 2/3 & 0 & 5/3 \\
 s_3 & 0 & 0 & -1/3 & -2/3 & 1 & -2/3 \\
 -z & 0 & 0 & -2/3 & -1/3 & 0 & -16/3
 \end{array}$$

Now we can use the dual simplex method to solve this tableau, getting:

$$\begin{array}{cccc|ccc}
 x & y & s_1 & s_2 & s_3 & \\
 \hline
 y & 0 & 1 & 3/2 & 0 & -1/2 & 1 \\
 x & 1 & 0 & -2 & 0 & 1 & 1 \\
 s_2 & 0 & 0 & 1/2 & 1 & -3/2 & 1 \\
 -z & 0 & 0 & -1/2 & 0 & -1/2 & -5
 \end{array}$$

In this example, we got the integer solution $(x, y) = (1, 1)$, so we are done. This is not always guaranteed to happen; it's possible that more fractional cuts will be necessary.

In general, the Gomory fractional cut can be stated as follows. Suppose that we have nonnegative integer variables $x_1, x_2, \dots, x_n \geq 0$ for which the following equation is valid:

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b.$$

Then we can deduce the inequality

$$\lfloor a_1 \rfloor x_1 + \lfloor a_2 \rfloor x_2 + \dots + \lfloor a_n \rfloor x_n \leq \lfloor b \rfloor$$

or, in the form that is more commonly used,

$$(a_1 - \lfloor a_1 \rfloor)x_1 + (a_2 - \lfloor a_2 \rfloor)x_2 + \dots + (a_n - \lfloor a_n \rfloor)x_n \geq b - \lfloor b \rfloor.$$

3 Extensions

The cutting plane method is often combined with the branch-and-bound method into a hybrid algorithm called “branch-and-cut”. Here, when solving a linear program and getting a fractional solution, we make a choice between two options:

- Pick a variable x_i with a fractional value, and use it to branch out to two new linear programs, as usual in the branch-and-bound method.
- Add a cutting plane inequality to replace the linear program by a new one with a different solution.

It's a matter of heuristics (in other words, guesswork) to decide between these two options. For example, we might add a fractional cut if it looks like the cut produced will be unusually strong. We might decide to branch if we've already tried several fractional cuts on the same linear program, and they haven't worked very well.