

Lecture 15: The Dual Simplex Method

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1 Dual feasible tableaux

Last time, we proved that if choosing basic variables \mathcal{B} in the primal gives an optimal solution, then $\mathbf{u}^\top = \mathbf{c}_\mathcal{B}^\top A_\mathcal{B}^{-1}$ is a dual optimal solution. One key fact from that proof was that if the row of reduced costs has the right signs for optimality, then \mathbf{u} is dual feasible.

More generally, define a tableau to be *dual feasible* if every reduced cost has the right sign for optimality. That is, when we're solving a minimization problem, a dual feasible tableau is one with nonnegative reduced costs. When we're solving a maximization problem, a dual feasible tableau is one with nonpositive reduced costs.

For other choices of \mathcal{B} , ones we haven't usually considered, the tableau can be dual feasible but not primal feasible: the rightmost column, which represents the values of the basic variables, might contain negative entries. In such cases, $\mathbf{u}^\top = \mathbf{c}_\mathcal{B}^\top A_\mathcal{B}^{-1}$ will produce a dual feasible solution, but it might not be dual optimal.

For an example of where we can find such a tableau, consider the following linear program (and its equational form on the right):

$$\begin{array}{ll}
 \underset{x,y \in \mathbb{R}}{\text{minimize}} & x + y \\
 \text{subject to} & 2x + y \geq 6 \\
 & 3x + y \geq 7 \\
 & x + 2y \geq 9 \\
 & x, y \geq 0
 \end{array}
 \quad \rightsquigarrow \quad
 \begin{array}{ll}
 \underset{x,y,s_1,s_2,s_3 \in \mathbb{R}}{\text{minimize}} & x + y \\
 \text{subject to} & -2x - y + s_1 = -6 \\
 & -3x - y + s_2 = -7 \\
 & -x - 2y + s_3 = -9 \\
 & x, y, s_1, s_2, s_3 \geq 0
 \end{array}$$

If we write down the tableau for $\mathcal{B} = (s_1, s_2, s_3)$, we get

	x	y	s_1	s_2	s_3	
s_1	-2	-1	1	0	0	-6
s_2	-3	-1	0	1	0	-7
s_3	-1	-2	0	0	1	-9
$-z$	1	1	0	0	0	0

This is not primal feasible, because s_1, s_2, s_3 are all negative. But it is dual feasible, because we're minimizing and all the reduced costs are nonnegative. We can summarize this state as having a tableau such that "if it were feasible, it would be optimal".

¹This document comes from the Math 482 course webpage: <https://faculty.math.illinois.edu/~mlavrov/courses/482-spring-2020.html>

The first nice thing about dual feasibility is that in many cases, a primal feasible basic solution is hard to find, but a dual feasible basic solution is easy. Here, we'd have to use the two-phase simplex method to find a basic feasible solution for the primal.

In principle, whenever we have a dual feasible tableau, we can use the formula $\mathbf{c}_B^\top A_B^{-1}$ to find a dual feasible solution, but we don't always want to. For the tableau above, the dual feasible solution is $(0, 0, 0)$.

2 The dual simplex method

The second nice thing about dual feasibility is that we can try to find an optimal solution by going from dual optimal tableau to dual optimal tableau. This is called the *dual simplex method*.

Consider the tableau above. It is dual feasible, but it is very far from primal feasible. How could we try to bring it closer to primal feasibility?

We can pick a row whose right-hand side is negative, such as the s_1 row in the example above. If we choose a negative entry in that row (the -2 in x 's column, or the -1 in y 's column) and make that the new pivot in that row, we'll divide by a negative number, and so the right-hand side will become positive.

Just as with the ordinary simplex method, not every choice is a good choice. In this case, there are two possible pivots we can choose in s_1 's row: x or y . Here are the consequences of those choices after row-reducing: x on the left, y on the right.

	x	y	s_1	s_2	s_3	
x	1	$1/2$	$-1/2$	0	0	3
s_2	0	$1/2$	$-3/2$	1	0	2
s_3	0	$-3/2$	$-1/2$	0	1	-6
$-z$	0	$1/2$	$1/2$	0	0	-3

	x	y	s_1	s_2	s_3	
y	2	1	-1	0	0	6
s_2	-1	0	-1	1	0	-1
s_3	3	0	-2	0	1	4
$-z$	-1	0	1	0	0	-6

Which is better? Well, both of these tableaus have made some progress in the primal: now two of the basic variables are positive. However, the second tableau (where we've pivoted on y) has stopped being dual feasible: one of the reduced costs is negative.

We want to avoid that, and preserve dual feasibility. To do that, we have to pick the pivot entry carefully. The choice of row was the leaving variable, and we were free to pick any leaving variable with a negative value. But if we want to preserve dual feasibility, the choice of entering variable is usually determined uniquely (apart from ties).

Suppose that once we chose the leaving variable, we are choosing the entering variable from the list x_1, x_2, \dots, x_k . Let a_1, a_2, \dots, a_k be their coefficients in the pivot row (all negative). Let r_1, r_2, \dots, r_k be their reduced costs (all the same sign, of whatever sign indicates dual feasibility).

If we pivot on x_i , then the new coefficients in the pivot row are

$$\left[\frac{a_1}{a_i} \quad \dots \quad \frac{a_{i-1}}{a_i} \quad 1 \quad \frac{a_{i+1}}{a_i} \quad \dots \quad \frac{a_k}{a_i} \right]$$

and when we row-reduce, the new reduced costs are

$$\left[r_1 - r_i \cdot \frac{a_1}{a_i} \quad \dots \quad r_{i-1} - r_i \cdot \frac{a_{i-1}}{a_i} \quad 0 \quad r_{i+1} - r_i \cdot \frac{a_{i+1}}{a_i} \quad \dots \quad r_k - r_i \cdot \frac{a_k}{a_i} \right].$$

The reduced cost of x_j goes from r_j to $r_j - r_i \cdot \frac{a_j}{a_i}$. We want the sign of this reduced cost to stay the same (whether we were minimizing or maximizing) so we want

$$|r_j| \geq \left| r_i \cdot \frac{a_j}{a_i} \right| \iff \left| \frac{r_j}{a_j} \right| \geq \left| \frac{r_i}{a_i} \right|.$$

This motivates a pivoting rule guaranteed not to change the sign of the reduced costs:

Among variables x_1, x_2, \dots, x_k with a negative coefficient in the pivot row, choose the one with the smallest absolute ratio $\left| \frac{r_i}{a_i} \right|$.

(The absolute values are there for generality. In fact, when we're minimizing, all of these ratios will be negative; when we're maximizing, all of these ratios will be positive.)

So we could have known, in our example, to pivot on x instead of y , because the ratio $\left| \frac{1}{-2} \right|$ is smaller than the ratio $\left| \frac{1}{-1} \right|$.

From the tableau with basis $\mathcal{B} = \{x, s_2, s_3\}$, we can do one more step. The leaving variable must be s_3 , because its value of -6 is the only negative one. The entering variable can be either y or s_1 . Their absolute ratios are $\left| \frac{1/2}{-3/2} \right| = \frac{1}{3}$ and $\left| \frac{1/2}{-1/2} \right| = 1$. Since y has the smallest ratio, we pivot on y , and get

	x	y	s_1	s_2	s_3	
x	1	0	$-2/3$	0	$1/3$	1
s_2	0	0	$-5/3$	1	$1/3$	0
y	0	1	$1/3$	0	$-2/3$	4
$-z$	0	0	$1/3$	0	$1/3$	-5

The resulting tableau has reached primal feasibility, while keeping dual feasibility. So now we're just at the usual kind of optimal tableau, and conclude that $(x, y) = (1, 4)$ is the optimal solution, with objective value 5.

In theory, what we did corresponded to using the simplex method on the dual linear program. But we never had to write down either the dual linear program or any dual solution to do it.

3 When can we use the dual simplex method?

3.1 Conveniently posed linear programs

We won't be able to do this for every linear program. The initial setup above required two things to be true:

- Starting off with a linear program composed only of inequalities, so that we add slack variables and can choose an initial basis consisting of slack variables only.
- Optimizing an objective function for which the starting tableau will be dual feasible. This can be either a minimization problem in which all costs are nonnegative (not too unusual) or a maximization problem in which all costs are nonpositive (this one's less common).

This provides an alternative to the primal simplex method, where we're off to an easy start if the right-hand-side vector \mathbf{b} is nonnegative. Now there are two things that can go right for us.

3.2 The two-phase dual simplex method

This is also something we can do in phase one of the two-phase simplex method. Here, our goal is just to find a basic feasible solution to begin with, and then we can continue with the simplex method as usual.

Instead of adding artificial variables to find a basic feasible solution, we can use the dual simplex method with an artificial objective function that has the properties described earlier. For instance, we might:

- Minimize 0. This is the simplest possible objective function with nonnegative costs. It will lead to degeneracy, though, and so we should worry about cycling.
- Minimize $x_1 + x_2 + \cdots + x_n$. This is also a fairly simple objective function with nonnegative costs.
- Minimize a function that we think will help us arrive at a good basic feasible solution. For example, if the actual objective function in the problem is “maximize $x - y$ ”, we could try the artificial objective function “minimize y ”, on the grounds that a small value of y will probably be better than a large one.

The full procedure is:

1. Come up with an artificial objective function.
2. Use the dual simplex method to minimize the artificial objective function.
3. Use the (ordinary) simplex method, starting from the final tableau of the previous step, to solve the original problem.