

Lecture 10: The Revised Simplex Method

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University of Illinois at Urbana-Champaign

1 Motivation for the revised simplex method

In the previous lecture, we've worked out the following formulas for the entries in a simplex tableau, once we've chosen the basis \mathcal{B} :

	$\mathbf{x}_{\mathcal{B}}$	$\mathbf{x}_{\mathcal{N}}$	
$\mathbf{x}_{\mathcal{B}}$	I	Q	\mathbf{p}
$-z$	$\mathbf{0}^T$	\mathbf{r}^T	$-z_0$

Where:

- $\mathbf{p} = A_{\mathcal{B}}^{-1}\mathbf{b}$.
- $Q = A_{\mathcal{B}}^{-1}A_{\mathcal{N}}$.
- $\mathbf{r}^T = \mathbf{c}_{\mathcal{N}}^T - \mathbf{c}_{\mathcal{B}}^T A_{\mathcal{B}}^{-1}A_{\mathcal{N}}$.

We can write this as $\mathbf{c}_{\mathcal{N}}^T - \mathbf{c}_{\mathcal{B}}^T Q$ if we already found Q , or as $\mathbf{c}_{\mathcal{N}}^T - \mathbf{u}^T A_{\mathcal{N}}$ in terms of $\mathbf{u}^T = \mathbf{c}_{\mathcal{B}}^T A_{\mathcal{B}}^{-1}$.

- $z_0 = \mathbf{c}_{\mathcal{B}}^T A_{\mathcal{B}}^{-1}\mathbf{b}$.

We can write this as $\mathbf{c}_{\mathcal{B}}^T \mathbf{p}$ if we already found \mathbf{p} , or as $\mathbf{u}^T \mathbf{b}$ in terms of $\mathbf{u}^T = \mathbf{c}_{\mathcal{B}}^T A_{\mathcal{B}}^{-1}$.

In the revised simplex method, we avoid recording, and row-reducing, the entire tableau. Instead, we only remember some of the most important values we'll need. The other values will be computed as we go, using the formulas above.

Which values do we need to remember?

The most important is the matrix $A_{\mathcal{B}}^{-1}$, because it shows up in every single formula. In theory, $A_{\mathcal{B}}^{-1}$ is *all* we will need.

In practice, we will also remember the vector $\mathbf{p} = A_{\mathcal{B}}^{-1}\mathbf{b}$, which tells us the values of the basic variables at the current basic feasible solution. Even though we could find it with a single matrix-vector multiplication, we keep it around, and update it by row reduction, because we know it will be necessary in every pivot step.

¹This document comes from the Math 482 course webpage: <https://faculty.math.illinois.edu/~mlavrov/courses/482-spring-2020.html>

2 The revised simplex method algorithm

So here's how we perform a step of the revised simplex method. Remember: at each step, we know only $A_{\mathcal{B}}^{-1}$, as well as $\mathbf{p} = A_{\mathcal{B}}^{-1}\mathbf{b}$. (Also, implicitly, we know \mathcal{B} and \mathcal{N} : we remember which variables are basic, and which variables are not basic.)

2.1 Pricing

Pricing refers to the step of the simplex method where we determine the entering variable.

When we had the entire tableau in front of us, this step was easy: just look for a reduced cost with the right sign (positive if we're maximizing, negative if we're minimizing) and pick that variable. When we don't have the reduced cost vector \mathbf{r}^T in front of us, that's harder.

Although we have a formula for \mathbf{r}^T using values we already know, we don't want to use it directly. We want to minimize the work we have to do!

Therefore, we will go through the nonbasic variables in \mathcal{N} , one at a time. For each nonbasic variable x_j , to decide if it's a valid entering variable, we compute its reduced cost r_j . Ideally, we use Bland's rule for pivoting: we pick the first nonbasic variable we can to be the entering variable. This means that we can stop as soon as we find the first reduced cost which has the right sign. (If we had a different way to choose the entering variable, such as picking the one with highest reduced cost, then we'd have to compute all of \mathbf{r}^T .)

How do we compute r_j ? We have a formula:

$$r_j = c_j - \mathbf{c}_{\mathcal{B}}^T A_{\mathcal{B}}^{-1} A_j.$$

Choosing the right order of operations is important here. We can begin evaluating this formula by multiplying either $\mathbf{c}_{\mathcal{B}}^T$ and $A_{\mathcal{B}}^{-1}$, or $A_{\mathcal{B}}^{-1}$ and A_j . This seems like a minor detail, but:

- If we find $A_{\mathcal{B}}^{-1}A_j$ first, that's a matrix-vector multiplication we must do for every single r_j . We'd like to avoid that.
- The product $\mathbf{c}_{\mathcal{B}}^T A_{\mathcal{B}}^{-1}$, which we gave the name \mathbf{u}^T , does not depend on which variable we're looking at. So we can compute it once at the beginning of the pricing step. Then, we can use the formula $r_j = c_j - \mathbf{u}^T A_j$, and this only requires multiplying together two vectors.

(For the computer scientists in the room: this is the difference between an $O(n^2)$ computation for every reduced cost r_j in the first case, and an $O(n)$ computation with a one-time $O(n^2)$ setup cost in the second case. If there are many nonbasic variables, the difference adds up!)

2.2 Column generation

Once we choose the entering variable x_j , our next step is to compute x_j 's column of the simplex tableau.

Again, we want to be as lazy as possible. We have a formula $Q = A_{\mathcal{B}}^{-1}A_{\mathcal{N}}$ for computing the entire simplex tableau (the nonbasic portion of it, that is). But we only need x_j 's column of Q . This is

$$Q_j = (A_{\mathcal{B}}^{-1}A_{\mathcal{N}})_j = A_{\mathcal{B}}^{-1}A_j.$$

(Yes, that's the multiplication we avoided doing in the previous step. We're going to do it now, but only one time: for the variable x_j that turned out to be the entering variable.)

2.3 Pivoting

We know the entering variable x_j , now it's time to choose the leaving variable. We already have all the inputs we need to make this decision: we know x_j 's column Q_j , and we know the right-hand side column \mathbf{p} .

As usual, our first step is to make a shortlist of possible leaving variables: from the basic variables \mathcal{B} , we pick out those variables with a positive entry in Q_j . That is, we consider the i^{th} basic variable as a possibility only if $Q_{ij} > 0$.

Among this shortlist, we settle things by computing ratios. For the i^{th} basic variable, the ratio is p_j/Q_{ij} . The basic variable on the shortlist with the smallest ratio is our leaving variable.

2.4 Updating $A_{\mathcal{B}}^{-1}$ and \mathbf{p}

The preceding steps have told us to go from the previous basis \mathcal{B} to a new basis \mathcal{B}' (removing some variable x_i and adding some variable x_j). Now, we need to know how to compute $A_{\mathcal{B}'}^{-1}$ and update \mathbf{p} to $A_{\mathcal{B}'}^{-1}\mathbf{b}$.

The rule for how to do this is simple:

The same row operations that we do to go from the old simplex tableau to the new one will turn $A_{\mathcal{B}}^{-1}$ to $A_{\mathcal{B}'}^{-1}$.

This seems a little bit too convenient. How do we know this will work?

Recall that we can think of $A_{\mathcal{B}}^{-1}$ as a summary of all the row operations that we did to our initial system of equations:

$$A\mathbf{x} = \mathbf{b} \quad \rightsquigarrow \quad A_{\mathcal{B}}^{-1}A\mathbf{x} = A_{\mathcal{B}}^{-1}\mathbf{b}.$$

When we perform one step of the simplex method and go from basis \mathcal{B} to \mathcal{B}' , we perform some extra row operations, summarized by some matrix M . So we've ended up at

$$A\mathbf{x} = \mathbf{b} \quad \rightsquigarrow \quad A_{\mathcal{B}}^{-1}A\mathbf{x} = A_{\mathcal{B}}^{-1}\mathbf{b} \quad \rightsquigarrow \quad MA_{\mathcal{B}}^{-1}A\mathbf{x} = MA_{\mathcal{B}}^{-1}\mathbf{b}.$$

But we also know that the matrix that puts the system of equations in row-reduced form for the basis \mathcal{B}' is just $A_{\mathcal{B}'}^{-1}$. So we've also ended up at

$$A\mathbf{x} = \mathbf{b} \quad \rightsquigarrow \quad A_{\mathcal{B}'}^{-1}A\mathbf{x} = A_{\mathcal{B}'}^{-1}\mathbf{b}.$$

These are just two ways of writing down the same final result, so they must be equal, and therefore $MA_{\mathcal{B}}^{-1} = A_{\mathcal{B}'}^{-1}$. That's just another way of saying that if we take the extra row operations represented by M , and apply them to $A_{\mathcal{B}}^{-1}$, we get $A_{\mathcal{B}'}^{-1}$.

There's a convenient way to represent this in a "mini-tableau" form. After we've determined our entering variable x_j , we can write down a smaller simplex tableau which only consists of the following information:

$$\left| A_{\mathcal{B}}^{-1} \mid Q_j \mid \mathbf{p} \right|$$

We can then row-reduce this mini-tableau. If the i^{th} basic variable is replaced by x_j , then we divide the i^{th} row by Q_{ij} , and then row-reduce to set all other entries of the x_j column to 0. This will have the side effect of updating $A_{\mathcal{B}}^{-1}$ and \mathbf{p} for the new basis.

3 An example

Consider the linear program

$$\begin{aligned} & \underset{x_1, x_2, x_3, x_4}{\text{maximize}} && x_1 + x_2 + x_3 + 2x_4 \\ & \text{subject to} && 3x_1 - x_2 + 4x_3 - x_4 \leq 4 \\ & && 2x_2 + x_4 \leq 5 \\ & && x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

or in equational form:

$$\begin{aligned} & \underset{x_1, x_2, x_3, x_4, s_1, s_2}{\text{maximize}} && x_1 + x_2 + x_3 + 2x_4 \\ & \text{subject to} && 3x_1 - x_2 + 4x_3 - x_4 + s_1 = 4 \\ & && 2x_2 + x_4 + s_2 = 5 \\ & && x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

3.1 Initialization

Our initial basis is $\mathcal{B} = (s_1, s_2)$ with

$$A_{\mathcal{B}}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{p} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

This is how things always work out when $\mathbf{x} = \mathbf{0}$ is a feasible solution, and the slack variables can be chosen as the initial basis. If that didn't happen, then we'd have to combine today's method with the two-phase simplex method.

3.2 Step 1

We begin by computing

$$\mathbf{u}^{\top} = \mathbf{c}_{\mathcal{B}}^{\top} A_{\mathcal{B}}^{-1} = [0 \ 0] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [0 \ 0].$$

Since this is just the zero vector, there's not much work to do: the formula $r_j = c_j - \mathbf{u}^{\top} \mathbf{A}_j$ simplifies to $r_j = c_j$. Before we've done any row reduction, the reduced costs are just the initial costs!

The very first nonbasic variable x_1 has a reduced cost of 1. We're maximizing, so it's a valid entering variable. The x_1 column of the tableau is computed by the formula $Q_j = A_{\mathcal{B}}^{-1} A_j$. Right now, $A_{\mathcal{B}}^{-1}$ is the identity matrix, and we just get $(3, 0)$: x_1 's column in the original matrix.

We write down the mini-tableau below on the left:

$$\begin{array}{c|cc|c|c} \mathcal{B} & A_{\mathcal{B}}^{-1} & x_1 & \mathbf{p} \\ \hline s_1 & \begin{bmatrix} 1 & 0 \end{bmatrix} & \begin{bmatrix} 3 \end{bmatrix} & \begin{bmatrix} 4 \end{bmatrix} \\ s_2 & \begin{bmatrix} 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 \end{bmatrix} & \begin{bmatrix} 5 \end{bmatrix} \end{array} \rightsquigarrow \begin{array}{c|cc|c|c} \mathcal{B} & A_{\mathcal{B}}^{-1} & x_1 & \mathbf{p} \\ \hline x_1 & \begin{bmatrix} 1/3 & 0 \end{bmatrix} & \begin{bmatrix} 1 \end{bmatrix} & \begin{bmatrix} 4/3 \end{bmatrix} \\ s_2 & \begin{bmatrix} 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 \end{bmatrix} & \begin{bmatrix} 5 \end{bmatrix} \end{array}$$

We pivot as in the ordinary simplex method; s_1 leaves the basis, and we row-reduce to get the tableau on the right.

3.3 Step 2

We begin by computing

$$\mathbf{u}^\top = \mathbf{c}_B^\top A_B^{-1} = [1 \ 0] \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} = [1/3 \ 0].$$

Apply the formula $r_j = c_j - \mathbf{u}^\top A_j$ to find the reduced costs, one at a time. The reduced cost of x_2 is

$$1 - [1/3 \ 0] [-1 \ 2] = \frac{4}{3} > 0,$$

so x_2 can enter the basis. Now we compute x_2 's column in the tableau: it is

$$A_B^{-1} A_2 = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2 \end{bmatrix}.$$

We write down the mini-tableau on the left below:

$$\begin{array}{c|cc|c} \mathcal{B} & A_B^{-1} & x_2 & \mathbf{p} \\ \hline x_1 & \begin{bmatrix} 1/3 & 0 \end{bmatrix} & \begin{bmatrix} -1/3 \end{bmatrix} & \begin{bmatrix} 4/3 \end{bmatrix} \\ s_2 & \begin{bmatrix} 0 & 1 \end{bmatrix} & \begin{bmatrix} 2 \end{bmatrix} & \begin{bmatrix} 5 \end{bmatrix} \end{array} \rightsquigarrow \begin{array}{c|cc|c} \mathcal{B} & A_B^{-1} & x_2 & \mathbf{p} \\ \hline x_1 & \begin{bmatrix} 1/3 & 1/6 \end{bmatrix} & \begin{bmatrix} 0 \end{bmatrix} & \begin{bmatrix} 13/6 \end{bmatrix} \\ x_2 & \begin{bmatrix} 0 & 1/2 \end{bmatrix} & \begin{bmatrix} 1 \end{bmatrix} & \begin{bmatrix} 5/2 \end{bmatrix} \end{array}$$

Only s_2 can leave the basis, so we row-reduce to replace it by x_2 . We get the tableau on the right.

3.4 Step 3

We begin by computing

$$\mathbf{u}^\top = \mathbf{c}_B^\top A_B^{-1} = [1 \ 1] \begin{bmatrix} 1/3 & 1/6 \\ 0 & 1/2 \end{bmatrix} = [1/3 \ 2/3].$$

Apply the formula $r_j = c_j - \mathbf{u}^\top A_j$ to find the reduced costs, one at a time. For x_3 , we get a reduced cost of

$$1 - [1/3 \ 2/3] \begin{bmatrix} 4 \\ 0 \end{bmatrix} = -\frac{1}{3} < 0,$$

so x_3 is not a valid entering variable. For x_4 , we get a reduced cost of

$$2 - [1/3 \ 2/3] \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{5}{3} > 0,$$

so x_4 is a valid entering variable. Now we compute x_4 's column in the tableau: it is

$$A_B^{-1} A_4 = \begin{bmatrix} 1/3 & 1/6 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/6 \\ 1/2 \end{bmatrix}.$$

We write down the mini-tableau on the left below; x_4 is forced to replace x_2 , since x_1 's coefficient is negative, and we row-reduce to get the tableau on the right.

$$\begin{array}{c|cc|c} \mathcal{B} & A_B^{-1} & x_4 & \mathbf{p} \\ \hline x_1 & \begin{bmatrix} 1/3 & 1/6 \end{bmatrix} & \begin{bmatrix} -1/6 \end{bmatrix} & \begin{bmatrix} 13/6 \end{bmatrix} \\ x_2 & \begin{bmatrix} 0 & 1/2 \end{bmatrix} & \begin{bmatrix} 1/2 \end{bmatrix} & \begin{bmatrix} 5/2 \end{bmatrix} \end{array} \rightsquigarrow \begin{array}{c|cc|c} \mathcal{B} & A_B^{-1} & x_4 & \mathbf{p} \\ \hline x_1 & \begin{bmatrix} 1/3 & 1/3 \end{bmatrix} & \begin{bmatrix} 0 \end{bmatrix} & \begin{bmatrix} 3 \end{bmatrix} \\ x_4 & \begin{bmatrix} 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 \end{bmatrix} & \begin{bmatrix} 5 \end{bmatrix} \end{array}$$

3.5 Step 4

We begin by computing

$$\mathbf{u}^\top = \mathbf{c}_B^\top A_B^{-1} = [1 \quad 2] \begin{bmatrix} 1/3 & 1/3 \\ 0 & 1 \end{bmatrix} = [1/3 \quad 7/3].$$

We could compute the reduced costs one at a time here, but it will turn out that this is the last step. So let's compute all of them by the formula

$$\mathbf{r}^\top = \mathbf{c}_N^\top - \mathbf{u}^\top A_N.$$

We get

$$[1 \quad 1 \quad 0 \quad 0] - [1/3 \quad 7/3] \begin{bmatrix} -1 & 4 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} = [-10/3 \quad -1/3 \quad -1/3 \quad -7/3].$$

All reduced costs are negative, so the current basic feasible solution is optimal.

We have $\mathbf{p} = (3, 5)$ which records the values of the basic variables x_1 and x_4 . This tells us that the optimal solution is

$$(x_1, x_2, x_3, x_4) = (3, 0, 0, 5).$$