1 What we know so far

Suppose we have a linear program \((P)\) with a dual \((D)\) as follows:

\[
\begin{align*}
(P) & \quad \begin{cases}
\text{maximize} & c^T x \\
\text{subject to} & Ax \leq b \\
x \geq 0
\end{cases} \\
(D) & \quad \begin{cases}
\text{minimize} & u^T b \\
\text{subject to} & u^T A \geq c^T \\
u \geq 0
\end{cases}
\end{align*}
\]

Let their optimal solutions be \(x^*\) and \(u^*\), respectively, with (by strong duality) equal objective value \(z^* = c^T x^* = u^*^T b\). Then we know that, for some \(\delta \in \mathbb{R}\) which is “sufficiently small” in absolute value,

- If we change \(c_i\) to \(c_i + \delta\), we expect \(x^*\) to remain primal optimal, and so \(z^*\) will change by \(\delta x_i\). This is guaranteed to be a lower bound.
- If we change \(b_i\) to \(b_i + \delta\), we expect \(u^*\) to remain dual optimal, and so \(z^*\) will change by \(\delta u_i\). This is guaranteed to be an upper bound.

(If we switch things up so that \((P)\) is a minimization problem and \((D)\) is a maximization problem, then the guarantees also switch: in that case, we’d get an upper bound in the first case and a lower bound in the second.)

We want to know: how large can \(\delta\) get before things go wrong, and the prediction is no longer exact?

To answer this, we will consider the example below, with the optimal tableau given to its right:

\[
\begin{array}{ccccccc}
\text{maximize} & 3x_1 - 2x_2 + 3x_3 \\
\text{subject to} & 2x_1 & -3x_3 & \leq & 1 \\
& 3x_1 & -x_2 & -x_3 & \leq & 2 \\
& x_2 & +2x_3 & \leq & 2 \\
x_1, x_2, x_3 & \geq & 0 \\
\end{array}
\]

\[
\begin{array}{ccccccc}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 \\
\hline
s_1 & 0 & 11/6 & 0 & 1 & -2/3 & 7/6 & 2 \\
x_1 & 1 & -1/6 & 0 & 0 & 1/3 & 1/6 & 1 \\
x_3 & 0 & 1/2 & 1 & 0 & 0 & 1/2 & 1 \\
-z & 0 & -3 & 0 & 0 & -1 & -2 & -6
\end{array}
\]

The optimal primal solution is \(x^* = (1, 0, 1)\), and the optimal dual solution is \(u^* = (0, 1, 2)\). These let us predict the rate of change in \(z^* = 6\) when \(c = (3, -2, 3)\) or \(b = (1, 2, 2)\) are changed; we just need to know how far we can trust these predictions.

\[\text{This document comes from the Math 482 course webpage: https://faculty.math.illinois.edu/~mlavrov/courses/482-fall-2019.html}\]
2 Easy cases

There are two cases in which a modification to the linear program is easy to analyze. They are exactly the cases in which the rules above predict no change at all.

2.1 Right-hand sides of slack constraints

Right now, $x^*$ satisfies $2x_1^* - 3x_3^* = -1 < 1$. So the constraint $2x_1 - 3x_3 \leq 1$ is slack.

If we increase the right-hand side of this constraint by any amount, the constraint will remain slack at $x^*$. This will not give us any new optimal solutions: if there were a better solution $x$ with a larger value of $2x_1 - 3x_3$, then we could improve $x^*$ by nudging it a little in the direction of $x$. In other words, this constraint isn’t limiting us anyway, so relaxing it or even dropping it isn’t going to change the optimal solution.

We can even decrease the right-hand side of this constraint by up to 2 (the current value of the corresponding slack variable $s_1$). In general, making the constraint tighter in this way could only hurt us, and decrease the objective value. But we know that $x^*$ would remain feasible if we made such change. So the objective value will stay the same.

This means that for any $\delta \in [-2, \infty)$, adding $\delta$ to the right-hand-side of this constraint keeps everything the same: $x^*$ remains optimal, and therefore the optimal objective value is still 6.

That was exactly what we expected based on looking at the dual solution, in which $u_1^* = 0$. We knew that for “sufficiently small” $\delta$, the solution should change by $\delta u_1^* = 0 \cdot \delta = 0$, and now we know that “sufficiently small” means “in the range $[-2, \infty)$”.

2.2 Costs of nonbasic variables

Right now, $x_2^* = 0$ in the optimal solution, so small changes to its coefficient $c_2 = -2$ shouldn’t affect the objective value. (In other words, the optimal objective value changes by $\delta x_2^* = 0 \cdot \delta = 0$.)

To analyze the magnitude of this change, we can consider the effect on the tableau. We never scale $-z$’s row, we only add and subtract from it. So adding $\delta$ to $c_2$ has the same effect in the optimal tableau: it changes to

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>0</td>
<td>$\frac{11}{6}$</td>
<td>0</td>
<td>1</td>
<td>$-\frac{2}{3}$</td>
<td>$\frac{7}{6}$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>$-\frac{1}{6}$</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{6}$</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$-z$</td>
<td>0</td>
<td>$\delta - 3$</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
<td>$-2$</td>
</tr>
</tbody>
</table>

For any $\delta \in (-\infty, 3]$, the reduced cost $\delta - 3$ will remain nonpositive, indicating that the tableau is still optimal. This means that we can change $c_2$ by any $\delta$ in that range, and the prediction that $z^*$ doesn’t change will be valid.

2.3 Summary

We have learned:
1. When \( s_t \) is basic (so the \( t \)th constraint is slack), we can change its right-hand side \( b_t \) to \( b_t + \delta \) for any \( \delta \in [-s_t^*, \infty) \) and keep the same optimal objective value.

2. When \( x_i \) is nonbasic, we can change its coefficient \( c_i \) by any \( \delta \in (-\infty, -r_i] \) (where \( r_i \) is the reduced cost of \( x_i \)) and keep the same optimal objective value.

(This rule assumes that we’re solving a maximization problem, and want to keep the reduced costs nonpositive. If we’re solving a minimization problem, we want to keep the reduced costs nonnegative, and the valid range is \( \delta \in [-r_i, \infty) \) instead.)

3 Hard cases

When the rules do predict a change in the objective value at some rate, it is still possible to say when that rate of change will remain valid, but it’s a bit harder.

3.1 Costs of basic variables

Suppose that we change \( c_1 \) to \( c_1 + \delta \). As before, we know that we can make this change in the optimal tableau:

\[
\begin{array}{cccccc}
  & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 \\
 s_1 & 0 & 11/6 & 0 & 1 & -2/3 & 7/6 \\
x_1 & 1 & -1/6 & 0 & 0 & 1/3 & 1/6 \\
x_3 & 0 & 1/2 & 1 & 0 & 0 & 1/2 \\
-z & -3 & \delta & 0 & 0 & -1 & -2 & -6
\end{array}
\]

Now, however, the tableau is not fully row-reduced. We must subtract \( \delta \) times \( x_1 \)'s row from \(-z\)'s row, getting

\[
\begin{array}{cccccc}
  & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 \\
 s_1 & 0 & 11/6 & 0 & 1 & -2/3 & 7/6 \\
x_1 & 1 & -1/6 & 0 & 0 & 1/3 & 1/6 \\
x_3 & 0 & 1/2 & 1 & 0 & 0 & 1/2 \\
-z & 0 & -3 + \frac{\delta}{6} & 0 & 0 & -1 - \frac{\delta}{3} & -2 - \frac{\delta}{6} & -6 - \delta
\end{array}
\]

Note that the new objective value here is \( 6 + \delta \), exactly as predicted by our rule: a change by \( \delta x_1^* = \delta \). However, that prediction is only valid if the new tableau above is still optimal: if the reduced costs are all nonnegative. This requires

\[
\begin{align*}
-3 + \frac{\delta}{6} & \leq 0 \\
-1 - \frac{\delta}{3} & \leq 0 \\
-2 - \frac{\delta}{6} & \leq 0
\end{align*} \quad \Rightarrow \quad \begin{cases} 
\delta \leq 18 \\
\delta \geq -3 \\
\delta \geq -12
\end{cases}
\]

All of these must be true, so we take the tightest upper and lower bound, and conclude that the allowable change is \( \delta \in [-3, 18] \).

In general, when modifying the coefficient \( c_i \), we consider every nonbasic variable \( x_j \) (or \( s_j \)) and compute the ratio

\[
\frac{\text{reduced cost of that variable}}{\text{coefficient of that variable in } x_i \text{'s row}}
\]
in the optimal tableau. Positive ratios give upper bounds and negative ratios give lower bounds.

(If you think about it, the same thing works whether the program is a minimization or a maximization problem.)

### 3.2 Right-hand sides of tight constraints

Suppose that we change the right-hand side $b_2$: the right-hand side of the constraint $3x_1 - x_2 - x_3 \leq 2$, which is tight for the current optimal solution $x^*$. The problem is that there is no $s_2$ row in the tableau, so we can’t just add $\delta$ to the right-hand side of the $s_2$ row and understand the effect of changing $b_2$ to $b_2 + \delta$.

However, the coefficients in $s_2$’s column represent how much of the original $s_2$ row contributed to a row of the optimal tableau. In other words, by looking at those coefficients, we know that changing $b_2$ from 2 to $2 + \delta$ will change the optimal tableau to the following:

<table>
<thead>
<tr>
<th></th>
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<th>$x_3$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
</tr>
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<tbody>
<tr>
<td>$s_1$</td>
<td>0</td>
<td>$11/6$</td>
<td>0</td>
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</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>$-1/6$</td>
<td>0</td>
<td>0</td>
<td>$1/3$</td>
<td>$1/6$</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>$1/2$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$1/2$</td>
</tr>
<tr>
<td>$-z$</td>
<td>0</td>
<td>$-3/6$</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
<td>$-2$</td>
</tr>
</tbody>
</table>

This predicts that the objective value will change from 6 to $6 + \delta$: good, that’s what our rule said. (It told us to expect a change by $\delta u_2^*$, and $u_2^* = 1$.)

This prediction will be valid for as long as the tableau above remains the optimal tableau. The reduced costs stay the same, but the values of the basic variables don’t: and if those values become negative, we lose primal feasibility, and need to update the tableau. So for the prediction to remain valid, we want to satisfy the conditions

$$\begin{align*}
2 - \frac{2}{3} \delta & \geq 0 \\
1 + \frac{1}{3} \delta & \geq 0 \\
1 + 0 \delta & \geq 0
\end{align*}$$

$$\Rightarrow \begin{cases} \delta \leq 3 \\
\delta \geq -3 \end{cases}$$

therefore any change $\delta \in [-3, 3]$ is fine.

In general, if we are considering changing $b_i$ to $b_i + \delta$, we should divide the right-hand side entries by the entries in $s_i$’s column (in the optimal tableau). Positive numbers will give us upper bounds on $\delta$ and negative numbers will give us lower bounds.