1 Finding the dual optimal solution

Suppose that we have a linear program (and a dual for it) as shown below:

\[
\begin{align*}
\text{(P)}: & \quad \begin{aligned}
\text{maximize} & \quad 2x + 3y \\
\text{subject to} & \quad -x + y \leq 3 \quad (u) \\
& \quad x - 2y \leq 2 \quad (v) \\
& \quad 3x + 4y \leq 26 \quad (w) \\
& \quad x, y \geq 0
\end{aligned} \\
\text{(D)}: & \quad \begin{aligned}
\text{minimize} & \quad 3u + 2v + 26w \\
\text{subject to} & \quad -u + v + 3w \geq 2 \quad (x) \\
& \quad u - 2v + 4w \geq 3 \quad (y) \\
& \quad u, v, w \geq 0
\end{aligned}
\end{align*}
\]

We can solve this linear program using the simplex method:

\[
\begin{array}{cccccc|cccccc}
& x & y & s_1 & s_2 & s_3 & \text{ } & x & y & s_1 & s_2 & s_3 \\
\hline
s_1 & -1 & 1 & 1 & 0 & 0 & 3 & \hline
s_2 & 1 & -2 & 0 & 1 & 0 & 2 & \sim & y & 0 & 1 & \frac{3}{7} & 0 & \frac{1}{7} & 5 \\
s_3 & 3 & 4 & 0 & 0 & 1 & 26 & \hline
\end{array}
\]

We can read off an optimal solution to \((P)\) from this pretty easily: it has \(x = 2\) and \(y = 5\). Can we also read off an optimal solution to \((D)\)?

We can! We’ll get there in several steps.

**Lemma 1.1.** If \(\mathcal{B}\) is an optimal basis for the primal program, then \(\mathbf{u}^\top = \mathbf{c}_\mathcal{B}^\top A_\mathcal{B}^{-1}\) is an optimal dual solution.

The formula in this lemma is motivated by complementary slackness. (But we won’t need the full power of complementary slackness, or of strong duality, to prove it—that’s important.) We can split up the dual constraints \(\mathbf{u}^\top A \geq \mathbf{c}^\top\) into two parts: \(\mathbf{u}^\top A_\mathcal{B} \geq \mathbf{c}_\mathcal{B}^\top\) and \(\mathbf{u}^\top A_\mathcal{N} \geq \mathbf{c}_\mathcal{N}^\top\). Generally (though not always) the basic variables in the primal optimal solution are going to be strictly positive. By complementary slackness, the corresponding constraints in the dual are going to be tight. This gives us the system of equations \(\mathbf{u}^\top A_\mathcal{B} = \mathbf{c}_\mathcal{B}^\top\), which we can solve for \(\mathbf{u}^\top\) by right-multiplying by \(A_\mathcal{B}^{-1}\). This gives the formula in the lemma.

**Proof.** But we prove that \(\mathbf{u}^\top = \mathbf{c}_\mathcal{B}^\top A_\mathcal{B}^{-1}\) is an optimal dual solution without using complementary slackness.
First, we check that it's feasible. The constraints \( u^T A_B \geq c_B^T \) hold because the right-hand side just simplifies to \((c_B^T A_B^{-1}) A_B = c_B^T\). More interesting are the constraints

\[
u^T A_N \geq c_N^T \iff c_B^T A_B^{-1} A_N \geq c_N^T \iff c_N^T - c_B^T A_B^{-1} A_N \leq 0^T.
\]

Why do these inequalities hold? Because we recognize \( c_N^T - c_B^T A_B^{-1} A_N \) as \( r^T \): the vector of nonbasic reduced costs in the optimal tableau. In an optimal solution, all the reduced costs should be nonnegative, which is just what the inequality says: so we've finished checking feasibility. (There are no nonnegativity constraints to check on \( u \) when \((P)\) is written in equational form.)

This is assuming we're solving a primal maximization problem. In a minimization problem, the reduced costs for an optimal solution will be nonpositive. However, the dual constraints will also point the other way for a minimization problem, so this works out.

Once we check feasibility, we should also check optimality, but that turns out to be quicker. The primal optimal solution has objective value

\[
c^T x = c_B^T x_B = c_B^T A_B^{-1} b
\]

(the first equation holds because all the nonbasic variables are 0, the second because of the formula for a basic solution). But the dual solution we've found also has objective value \( u^T b = c_B^T A_B^{-1} b \). When a primal feasible solution and a dual feasible solution have the same objective value, they must both be optimal—and it takes only weak duality to prove that.

Okay, so we have the formula \( u^T = c_B^T A_B^{-1} \). How do we use it?

In the general case, this is kind of tricky. We have to know \( A_B^{-1} \), which doesn’t necessarily appear in the tableau. (We do know it if we’re using the revised simplex method! With the revised simplex method, using this formula is easy, and in fact, we may have already computed \( c_B^T A_B^{-1} \) in the process of finding the optimal primal solution.)

However, one case where we do know \( A_B^{-1} \) is when we added slack variables to the linear program, as in the example above. If so, then the initial tableau contains an identity matrix in the slack variable columns. The row-reduction steps that go from the initial tableau to the final tableau are exactly equivalent to multiplying by \( A_B^{-1} \). Therefore the slack variable columns of the tableau contain \( A_B^{-1} \) times \( I \), which is just \( c_B^T A_B^{-1} \).

This gives one way (not the best way, but we can improve on it) of finding an optimal dual solution in the example. Take \( c_B^T \) from the objective function and \( A_B^{-1} \) from the final tableau, and we get

\[
[u \ v \ w] = c_B^T A_B^{-1} = \begin{bmatrix} 3/7 & 0 & 1/7 \\ 10/7 & 1 & 1/7 \\ -4/7 & 0 & 1/7 \end{bmatrix} = \begin{bmatrix} 1/7 & 0 & 5/7 \end{bmatrix}.
\]

But we can find this solution even faster. For any group of variables \( S \), their reduced costs are given by the formula

\[
c_S^T - c_B^T A_B^{-1} A_S.
\]

Mostly we’ve been using this when \( S = N \), the set of nonbasic variables, because those reduced costs are typically the most interesting ones. But now, take \( S \) to be the set of slack variables.
Then $c_S^T = 0^T$ (the slack variables don’t appear in the objective function) and $A_S = I$ (the slack variable portion of the constraint matrix is just the identity matrix), so their reduced costs simplify to

$$-c_B^T A_B^{-1}$$

or just the negative of the optimal dual solution we’re looking for! So we can get $\begin{bmatrix} 1/7 & 0 & 5/7 \end{bmatrix}$ without doing any arithmetic, just by looking at the reduced costs of the slack variables.

## 2 Strong duality

We now have a formula for an optimal dual solution with the same objective value as the optimal primal solution. This is a proof of strong duality: that the primal and dual linear programs have the same optimal objective value!

(This is somehow not a very well-respected proof, because in order for it to work, we need to know that the simplex method always works, which has its own convoluted proof. But we’ve already done all that hard work, so we might as well use it.)

Specifically:

- From the lemma in the previous section, we show that whenever the primal program has an optimal solution, so does the dual (because we found one) and the objective values agree (because that’s how we proved it was optimal).
- Because duality is symmetric, we get the converse for free: whenever the dual program has an optimal solution, so does the primal, and the objective values agree.
- Our proof only works for linear programs in equational form, but we can put all linear programs in this form, and that plays well with duality.

As a bonus, we can learn what happens when the primal or the dual program is infeasible or unbounded. Specifically, there are only four possible cases:

1. Both the primal program and the dual program have an optimal solution. (Most of the examples we’ve seen have been like this.)
2. The primal is infeasible, and the dual is unbounded.
3. The primal is unbounded, and the dual is infeasible.
4. Both the primal and the dual are infeasible.

Why can’t other things happen? Well, there’s two arguments ruling out the other cases.

First, the primal program and the dual program can’t both be unbounded. If the primal is unbounded, that means in particular that it has a feasible solution. The objective value of that feasible solution limits the possible objective values in the dual, so the dual can’t be unbounded.

Second, strong duality says that if one of the two programs has an optimal solution, the other can’t be infeasible or unbounded.
Here are some examples of the cases that are possible. In the pair

\[(P) \begin{cases} \text{maximize} & x \\ x \in \mathbb{R} \\ \text{subject to} & x \leq -1 \ (u) \\ & x \geq 0 \end{cases} \quad (D) \begin{cases} \text{minimize} & -u \\ u \in \mathbb{R} \\ \text{subject to} & u \geq 1 \ (x) \\ & u \geq 0 \end{cases}\]

the primal program is infeasible (we can’t have $x \leq -1$ and $x \geq 0$ at the same time) and the dual program is unbounded (by setting $u$ to be very large, we make $-u$ very small).

We can get an example where the primal program is infeasible and the dual program is unbounded simply by reversing the roles of the two programs. Or, if we want to keep things as a max and a min, we could do a slight variant of the example above:

\[(P) \begin{cases} \text{maximize} & y \\ y \in \mathbb{R} \\ \text{subject to} & -y \leq 1 \ (v) \\ & y \geq 0 \end{cases} \quad (D) \begin{cases} \text{minimize} & v \\ v \in \mathbb{R} \\ \text{subject to} & -v \geq 1 \ (y) \\ & v \geq 0 \end{cases}\]

Here, any nonnegative $y$ is primal feasible, but no $v$ is dual feasible.

To get an example where both linear programs are infeasible, just take these two examples and stick them together:

\[(P) \begin{cases} \text{maximize} & x + y \\ x, y \in \mathbb{R} \\ \text{subject to} & x \leq -1 \ (u) \\ & -y \leq 1 \ (v) \\ & x, y \geq 0 \end{cases} \quad (D) \begin{cases} \text{minimize} & -u + v \\ u, v \in \mathbb{R} \\ \text{subject to} & u \geq 1 \ (x) \\ & -v \geq 1 \ (y) \\ & u, v \geq 0 \end{cases}\]

Here, the primal is infeasible because we can’t choose a value of $x$, and the dual is infeasible because we can’t choose a value of $v$. 