1. Use the method described at the end of the notes for Lecture 29 (Push-Relabel II) to initialize the push-relabel algorithm for the network below.

```
 s  4  b  2  c  5  t
  5 1  2
  3 3  4
 d
```

Then, perform all possible push steps that can be done without having to relabel a node. (You don’t have to complete the algorithm from there.)

2. Suppose you use the push-relabel algorithm to find a maximum-flow in the very boring $n$-node network below:

```
 s  n-1  v_1  n-2  v_2  n-3  \cdots  2  v_{n-2}  1  t
```

(In other words: the nodes are labeled $s, v_1, v_2, \ldots, v_{n-2}, t$ in that order, with an arc from every node to the next. The capacity of the first arc is $n-1$, and each arc after that has capacity 1 smaller than the previous, so that the the last arc has a capacity of 1.)

Describe, in words, how the algorithm will go. Give the labels (heights) of each node at the conclusion of the push-relabel algorithm.

3. Suppose that we are using the primal-dual method to solve the linear program

```
\begin{align*}
\text{minimize} & \quad 3x_1 + x_2 + x_4 \\
\text{subject to} & \quad x_1 + 2x_2 - 2x_3 - x_4 = 2 \\
& \quad x_1 + x_2 - 2x_4 = 3 \\
& \quad x_1 - 2x_2 - x_3 + 2x_4 = 4 \\
& \quad x_1, x_2, x_3, x_4 \geq 0
\end{align*}
```

and that we are currently at the dual solution $u = (1, 1, 1)$.

(a) Write down and solve (RP): the restricted primal program.

(b) Find the optimal solution $v$ to (DRP): the dual of the restricted primal.

(c) Use $v$ to find an improved dual solution to the original linear program.
(d) Show that the new dual solution is optimal by finding a corresponding optimal primal solution.

4. Consider the linear program

\[
\begin{align*}
\text{minimize} & \quad x_1 + 3x_2 - 2x_3 - x_4 \\
\text{subject to} & \quad x_1 + 2x_2 - 2x_3 - x_4 = 2 \\
& \quad x_1 + x_2 - 2x_4 = 3 \\
& \quad x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

(a) Use the assumption that \(x_1 + x_2 + x_3 + x_4 \leq 100\) in all feasible solutions to the linear program above, to write down an equivalent linear program and a feasible dual solution for it.

(b) Perform two steps of the primal-dual method, starting from the feasible dual solution you found.

5. (Only 4-credit students need to do this problem.)

Suppose that you have a network in which, instead of every arc having a capacity, there is a capacity associated through every node (other than the source \(s\) or the sink \(t\)). The flow along an arc can be arbitrary, but the total flow going into a node (equivalently, the total flow going out of a node) can be at most the capacity of that node.

Explain how to convert a maximum-flow problem for such a network into a standard maximum-flow problem.