1. A zero-sum game between Alice and Bob has the payoff matrix below (with respect to Alice). Determine the possible saddle points this game could have, depending on the value of $x$, and for which $x$ those outcomes would be saddle points.

$$
\begin{array}{c|ccc}
\text{Alice: } & \text{Bob: } a & \text{Bob: } b & \text{Bob: } c \\
\hline
\text{Alice: } A & 2 & 3 & 8 \\
\text{Alice: } B & 5 & x & 7 \\
\text{Alice: } C & 4 & 1 & 0 \\
\end{array}
$$

2. Alice and Bob each have a nickel (5 cents), a dime (10 cents), and a quarter (25 cents). They simultaneously put a coin down on the table. If the coins are equal in value, Alice wins Bob’s coin; if Alice’s coin is more valuable, Bob wins Alice’s coin; if Bob’s coin is more valuable, nothing happens.

Draw a payoff matrix for this game and write down a linear program by which Alice could determine her optimal strategy.

3. Solve the linear program below using the revised simplex method:

$$
\begin{align*}
\text{maximize} & \quad x + y \\
\text{subject to} & \quad -x + 3y \leq 6 \\
& \quad 2x - y \leq 8 \\
& \quad x, y \geq 0 \\
\end{align*}
$$

4. Use Fourier–Motzkin elimination to find a point satisfying

$$
\begin{align*}
x + y - z & \leq 5 \\
2x - y + 2z & \leq -2 \\
x + 2y & \leq -1 \\
-3x - y + 2z & \leq 1 \\
\end{align*}
$$

5. (Only 4-credit students need to do this problem.)

Use Farkas’s lemma to prove LP duality in the following form: if the linear program (P) below cannot achieve an objective value of at least $z^*$, and the dual program (D) is feasible, then the dual linear program (D) has a feasible solution $u$ with objective value less than $z^*$.

$$(P) \begin{cases}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad A x \leq b \\
\end{cases} \quad (D) \begin{cases}
\text{minimize} & \quad u^T b \\
\text{subject to} & \quad u^T A = c^T \\
& \quad u \geq 0 \\
\end{cases}$$