1. In the video game Minecraft, players can plant sugar cane. A sugar cane farm is a rectangular grid of blocks, where each block is either water or dirt. If a dirt block shares an edge with at least one water block, then a sugar cane plant can be grown on that dirt block. (Sugar cane plants cannot be grown on water blocks.)

Write an integer linear program to maximize the number of sugar cane plants which can be grown on an $n \times n$ sugar cane farm, by arranging the water blocks appropriately. There will be many similar constraints, so it’s fine if you write out in words something like “we include [constraint] for every block in the grid, except for [some cases], in which we include [constraint] instead”.

(Hint: use variables $x_{ij}$ to indicate whether a block is a water block or a dirt block, and $y_{ij}$ to indicate whether a sugar cane plant can grow on a block.)

2. Solve the integer linear program below using the branch-and-bound method.

$$\begin{align*}
\text{maximize} & \quad 2x + y \\
\text{subject to} & \quad -x + y \leq 0 \\
& \quad 6x + 2y \leq 21 \\
& \quad x, y \geq 0
\end{align*}$$

(Note: in some subproblems, you may be able to solve an LP by looking at it, without using the simplex algorithm. If you do, that’s fine. But I do want you to write down, at each step of the branch-and-bound method, which LP you solve, what the optimal solution is, and what its objective value is.)

3. Solve the integer linear program from problem 2 again, this time using fractional cuts.

4. (Only 4-credit students need to do this problem.)

Suppose we are solving an integer linear program

$$\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b \\
& \quad x \geq 0
\end{align*}$$

for which the branch-and-bound algorithm is going to take too long to find the optimal answer (call this unknown optimal answer $x^\star$). However, we’re willing to settle for an approximate solution. Specifically, we want a 2-approximation: a feasible (integer) solution $x \in \mathbb{Z}^n$ such that $c^T x \geq \frac{1}{2} c^T x^\star$. (Assume that $c \geq 0$, so that none of these objective values are negative.)

Describe a modification of the branch-and-bound algorithm that finds a 2-approximation (and branches less often as a result).