Infinite Loops in the Ford–Fulkerson Algorithm

Here is the example I didn’t finish in class in which augmenting along the wrong paths could potentially lead us in an infinite loop.

I get this example by way of the notes at http://jeffe.cs.illinois.edu/teaching/algorithms/2009/notes/16-maxflowalgs.pdf, which use an example originally given in “The smallest nets-works on which the Ford–Fulkerson maximum flow procedure may fail to terminate” by Uri Zwick. Feel free to consult either of these sources, but I hope to develop the example in more detail here.

Consider the network

\[
\begin{array}{ccc}
  & s & \\
 10 & a & b \\
1 & b & c \\
10 & t & d \\
\end{array}
\]

where the capacity \( x \) is the real number \( \frac{\sqrt{5}-1}{2} \approx 0.618 \). This value is chosen to satisfy \( x^2 = 1 - x \).

A maximum flow in this network has value 21, but we’ll never get there: we’ll be stuck at less than 6.24 forever. So for the purposes of our augmenting paths, the edges with capacity 10 might as well have infinite capacity; we’ll never use it all up anyway.

We begin with the zero flow and perform the following augmenting steps. Diagrams on the left will show the augmenting path only, and diagrams on the right will show only the nonzero flows on the edges. For the capacities, refer to the diagram above.

1. Augment along the path \( s \to b \to c \to t \) (shown on the left), getting a flow with a value of 1 (shown on the right). The bottleneck is the edge \( b \to c \), which has a total capacity of 1.
2. Augment along the path $s \rightarrow d \rightarrow c \leftarrow b \rightarrow a \rightarrow t$ (shown on the left), getting a flow with a value of $1 + x$ (shown on the right). The bottleneck is the edge $d \rightarrow c$, which has a total capacity of $x \approx 0.618$.

3. Augment along the path $s \rightarrow b \rightarrow c \leftarrow d \rightarrow t$ (shown on the left), getting a flow with a value of $1 + 2x$ (shown on the right). The bottlenecks are the edge $b \rightarrow c$ (which has total capacity of 1, with $1 - x$ used) and the backward edge $c \rightarrow d$ (which has a flow of $x$, so it can be reduced by at most $x$).

4. Augment along the path $s \rightarrow d \rightarrow c \leftarrow b \rightarrow a \rightarrow t$ (shown on the left), increasing the value by $x^2$ and getting a flow with a value of $1 + 2x + x^2$ (shown on the right). The bottleneck is the edge $b \rightarrow a$, which has total capacity 1, with $\approx 0.618$ used, and $x^2 \approx 0.382$ capacity to spare.
5. Augment along the path \( s \rightarrow a \leftarrow b \rightarrow c \rightarrow t \) (shown on the left), increasing the value by \( x^2 \) and getting a flow with a value of \( 1 + 2x + 2x^2 = 3 \) (shown on the right). The bottleneck is the edge \( b \rightarrow c \), which has total capacity 1, with \( \approx 0.618 \) used, so it has \( x^2 \approx 0.382 \) capacity to spare.

From here, the pattern will be to repeat the augmenting paths used in steps 2–5 over and over again. In this case, the total value they contributed was \( x + x + x^2 + x^2 \). In the next iteration, they will only contribute \( x^3 + x^3 + x^4 + x^4 \), and then \( x^5 + x^5 + x^6 + x^6 \), and so on. The total value obtained in this way will be

\[
1 + 2x + 2x^2 + 2x^3 + 2x^4 + \cdots = 1 + \frac{2}{1 - x} = 1 + \frac{4}{3 - \sqrt{5}} = 4 + \sqrt{5} \approx 6.236,
\]

but getting there will take infinitely many steps, and that isn’t even close to the maximum possible. I will repeat one more iteration (steps 6–9) so you can see the pattern.

6. Augment along the path \( s \rightarrow d \rightarrow c \leftarrow b \rightarrow a \rightarrow t \) (shown on the left), getting a flow with a value of \( 3 + x^3 \) (shown on the right). The bottleneck is the edge \( d \rightarrow c \), which has a total capacity of \( \approx 0.618 \), with \( \approx 0.382 \) used, so it has \( x^3 \approx 0.236 \) capacity to spare.
7. Augment along the path $s \rightarrow b \rightarrow c \leftarrow d \rightarrow t$ (shown on the left), getting a flow with a value of $3 + 2x^3$ (shown on the right). The bottleneck is the edge $b \rightarrow c$ (which has total capacity of 1, with $\approx 0.764$ used and $x^3 \approx 0.236$ to spare).

8. Augment along the path $s \rightarrow d \rightarrow c \leftarrow b \rightarrow a \rightarrow t$ (shown on the left), increasing the value by $x^4$ and getting a flow with a value of $3 + 2x^3 + x^4$ (shown on the right). The bottleneck is the edge $b \rightarrow a$, which has total capacity 1, with $\approx 0.854$ used, so it has $x^4 \approx 0.146$ capacity to spare.

9. Augment along the path $s \rightarrow a \leftarrow b \rightarrow c \rightarrow t$ (shown on the left), increasing the value by $x^4$ and getting a flow with a value of $3 + 2x^3 + 2x^4$ (shown on the right). The bottleneck is the edge $b \rightarrow c$, which has total capacity 1, with $\approx 0.854$ used, so it has $x^4 \approx 0.146$ capacity to spare.