1. Let $A$ be an $m \times n$ binary matrix (a matrix whose elements are all either 0 or 1). A line in $A$ is a row or a column. We consider two 1’s in $A$ independent if no line contains both of them. Prove that the maximum number of pairwise independent 1’s is equal to the minimum number of lines that cover all the 1’s in $A$.

(Hint: use a bipartite graph related to $A$.)

2. Let $D$ a directed graph. For a set $S$ of vertices of $D$, let $N^+(S)$ denote the “out-neighborhood” of $D$: all the endpoints $y$ of directed edges $(x, y)$ with $x \in S$. (Note that $N^+(S)$ may intersect $S$, if there are edges from one vertex in $S$ to another. It is even possible for $N^+(S)$ to contain $S$.)

Prove that there exist pairwise disjoint cycles in $D$ such that each vertex of $D$ lies in exactly one of the cycles if and only if $|N^+(S)| \geq |S|$ for all $S \subseteq V(D)$.

3. Determine the two possible stable matchings between $\{v, w, x, y, z\}$ and $\{a, b, c, d, e\}$ we get by performing the Gale–Shapley algorithm with the preference lists below: first with $\{v, w, x, y, z\}$ making offers, and second with $\{a, b, c, d, e\}$ making offers.

$$
\begin{align*}
v : & c > d > b > e > a \\
w : & a > c > e > d > b \\
x : & a > e > d > c > b \\
y : & b > d > e > a > c \\
z : & b > a > c > e > d
\end{align*}
\begin{align*}
a : & v > y > z > x > w \\
b : & x > w > v > z > y \\
c : & x > y > z > w > v \\
d : & x > z > v > w > y \\
e : & x > y > w > z > v
\end{align*}
$$

4. Construct a 5-regular simple graph that (i) has no perfect matching and (ii) remains connected after deleting any 2 edges.

5. (4-credit students only)

Suppose that we are performing the Gale–Shapley algorithm to find a stable matching between sets $X$ and $Y$, with side $X$ making the offers.

Prove that if there exists any stable matching where $x \in X$ is matched to $y \in Y$, then $y$ does not reject $x$ in the course of performing the algorithm. (That is, either $y$ never evaluates $x$, or $y$ always accepts $x$’s offer over all others, and ends up matched with $x$.)

(Hint: arguing by contradiction, consider the first time such a rejection occurs.)