1. Prove that if $G$ is a connected graph with exactly 4 vertices of odd degree, there exist two trails in $G$ such that each edge is in exactly one trail. Find a graph with 4 vertices of odd degree that’s not connected for which this isn’t true.

2. Given a positive integer $k$, let $G(k)$ be the graph whose vertex set consists of all subsets of \{1, 2, \ldots, 2k + 1\} which have size $k$ or $k + 1$, and in which two vertices are adjacent if there is exactly one element present in one set but not the other. For example, $G(1)$ is the graph

\[
\begin{align*}
\{1, 2\} & \rightarrow \{2\} \\
\{1\} & \rightarrow \{2, 3\} \\
\{1, 3\} & \rightarrow \{3\}
\end{align*}
\]

(a) Prove that $G(k)$ is regular and bipartite for all $k$.

(b) Compute the length of a shortest cycle of $G(k)$ (this number is called the girth of a graph).

3. Let $G$ be a graph on $n \geq 2$ vertices with maximum degree $\Delta(G)$. Prove or disprove: if a vertex of degree $\Delta(G)$ is deleted, then the average degree of $G$ must decrease.

4. Show that for all even $n$, and for all $r$ such that $0 \leq r \leq n - 1$, there exists an $r$-regular graph with $n$ vertices.

(Try to come up with a symmetric construction. If every vertex looks the same, it’s much easier to prove that the construction works!)

5. (4-credit students only) Let $A_n$ be the graph whose vertex set is $\{0, 1\}^n$ (it consists of all $n$-tuples whose entries are 0 or 1) and in which two vertices are adjacent if either

- they differ only in the $1^{\text{st}}$ entry (for example, $(0, 0, 1)$ and $(1, 0, 1)$ are adjacent for $n = 3$), or
- they differ only in the $k^{\text{th}}$ entry, and their first $k - 1$ entries are all 0 (for example, $(0, 0, 1)$ and $(0, 0, 0)$ are adjacent for $n = 3$).

Prove that $A_n$ is connected for all $n$. 