7. Some aspects of topology

Definition 7.1. *A topological space is given by a tuple* $(X, \mathcal{O})$ *where* X *is a set* $\mathcal{O} \subset P(X)$ *is a collection of set such that*

i) $\emptyset \in \mathcal{O}$,

ii) *If* $O_1, O_2 \in \mathcal{O}$, *then* $O_1 \cap O_2 \in \mathcal{O}$,

iii) *If* $(O_i)_i \subset \mathcal{O}$, *then* $\bigcup_i O_i \in \mathcal{O}$.

A basis $\mathcal{B}$ *of a topology is a collection of set such that* $O \in \mathcal{O}$ *if and only if for every* $x \in O$ *there are* $B_1, ..., B_m \in \mathcal{B}$ *such that*

$$B_1 \cap \cdots \cap B_m \in O \, .$$

Example 7.2. *Let* (X, d) *be a metric space. Then*

$$\mathcal{O} \;=\; \{O \subset X \;:\; O \text{ open}\}$$

defines a topology which is Hausdorff, i.e. for $x \neq y$ *we find* O_x *and* O_y *in* $\mathcal{O}$ *such that*

$$\emptyset \;=\; O_x \cap O_y \, .$$

Example 7.3. *Let* $X \;=\; \mathbb{R} \cup \{\infty\}$. *Then*

$$\mathcal{O} \;=\; \{O \;:\; O \subset \mathbb{R} \;O \text{ open}\} \cup \{K^c \cup \{\infty\} \;:\; K \;K \text{ compact }\}$$

defines a topology.

Definition 7.4. *A function* $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *is continuous if*

$$f^{-1}(O) \in \mathcal{O}_X$$

for all $O \in \mathcal{O}_Y$. *A sequence* (x_n) *converges to* x *if for every open set* $O \in \mathcal{O}$ *containing* x *there exists* n_0 *such that*

$$x_n \in O$$

holds for all $n > n_0$. *The definition of compactness is the same as for metric spaces.*

Remark 7.5. *There exists a Let* $S^1 = \{z \in \mathbb{C} \;:\; |z| = 1\}$. *The map* $\phi : S^1 \to \mathbb{R} \cup \{\infty\}$ *defined by* $\phi(e^{2\pi i t}) = \frac{\sin(\pi t)}{\cos(\pi t)}$ *yields a homeomorphism between* S^1 *and* $\mathbb{R} \cup \infty$, *i.e.* ϕ *is bijective and* $\phi(O)$ *open if and only if* O *is open.*

Example 7.6. *Let* $X \;=\; \{(x_n) \;:\; x_n \in \mathbb{R}\}$. *Define*

$$d((x_n), (y_n)) \;=\; \sum_n 2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|} \, .$$

Then $O \subset X$ is open if and only if for every $(x_n) \in O$ there exists n_0 and $\varepsilon > 0$ such that if

$$|y_n - x_n| < \varepsilon$$

holds for all $n \leq n_0$, then $(y_n) \in O$.

Let X be a set. On the space $F(X, \mathbb{R}) = \mathbb{R}^X$ of arbitrary functions, we define for $x \in X$ and $O \subset \mathbb{R}$

$$B_x^O \ = \ \{g \ : \ g(x) \in O\} \, .$$

Let $\mathcal{O}$ be defined as the collection of sets such that $O \in \mathcal{O}$ of and only if for every $f \in O$ there exists $x_1, ..., x_m$ and O_1,...,O_m such that

$$f \in B_{x_1}^{O_1} \cap \cdots B_{x_m}^{O_m} \subset O \, .$$

PROPOSITION 7.7. *Let $f \in F(X, \mathbb{R})$ and (f_n) be a sequence if functions. Then* $\lim_n f_n = f$ *with respect to the topology above if and only of* $\lim_n f_n(x) = f(x)$ *holds for all $x \in X$.*

PROOF. Let $x \in X$. We define $\delta_x : F(X, \mathbb{R}) \to \mathbb{R}$ by $\delta_x(f) = f(x)$. By definition we see that δ_x is continuous. Thus $f = \lim_n f_n$ implies

$$f(x) \ = \ \delta_x(f) \ = \ \lim_n \delta_x(f_n) \ = \ \lim_n f_n(x) \, .$$

Conversely, we assume that $\lim_n f_n(x) = f(x)$ for all $x \in \mathbb{R}$. Let O be an open set such that $f \in O$. By definition there are $x_1, ..., x_m \in X$ and $O_1, ..., O_m$ subsets of $\mathbb{R}$ such that

$$f \in B_{x_1}^{O_1} \cap \cdots \cap B_{x_m}^{O_m} \subset O \, .$$

Thus we may choose n_0 such that

$$f_n(x_i) \in O_i$$

for all $i = 1, ..., m$ and $n > n_0$. This implies

$$f_n \in O$$

for all $n > n_0$. ■

THEOREM 7.8. *(Tychonov) Let $C \subset F(\mathbb{R})$ be a closed set. Then C is compact if and only if all the sets $\delta_x(C)$ are compact.*

We need a little preparation. We say that a collection (F_i) has the finite intersection property (fip) if

$$\emptyset \neq \cap_{S \subset I} F_i$$

holds for every finite subset $S \subset I$.

LEMMA 7.9. *Let $C \subset X$ be a set. Then C is compact if for every family (F_i) of closed sets such that $(C \cap F_i)$ has fip, then*

$$\emptyset \neq C \cap \bigcap_i F_i \, .$$

PROOF. Homework ■

Theorem. Since δ_x is continuous, we see that for compact C we must have $\delta_x(C)$ compact. The converse is more involved. The shortest proof I know uses ultrafilters. We skip it. ■

DEFINITION 7.10. *A set $E \subset X$ of a topological space is called connected, if for all open sets O_1 and O_2*

$$E \subset O_1 \cup O_2 \quad \textit{and} \quad E \cap O_1 \cap O_2 \, = \, \emptyset$$

implies $E \subset O_1$ or $E \subset O_2$.

LEMMA 7.11. *A subset E in $\mathbb{R}$ is connected if and only if E is an interval.*

PROOF. I is called an interval if $a < c < b$ and $a, b \in I$ implies $c \in I$. If E is not an interval we find $a, b \in I$ and $a < c < b$ such that $c \notin I$. We define $O_1 = (-\infty, c)$ and $O_2 = (c, \infty)$. The condition for connectedness are then violated. Conversely, we assume that I is an interval. Let O_1 and O_2 be open sets such that $I \subset O_1 \cup O_2$ and $O_1 \cap I \neq \emptyset$ and $O_2 \cap I \neq \emptyset$. We may assume that $a < b$ and $a \in O_1$, $b \in O_2$. We define

$$c \, = \, \sup\{t \, : a \leq t \leq b, \, \forall_{a \leq s \leq t} s \in O_1\} \, .$$

If $c = b$, then $O_1 \cap O_2 \neq \emptyset$. If $c < b$, then $c \notin O_1$ because if it were we could use the fact that O_1 is open and find $\delta > 0$ such that $c + \delta$ satisfies the conditions and $c + \delta \leq c$. The same argument shows that $c \neq a$. Thus $c \in O_2$. Since O_2 is open we known that there exists a $\delta > 0$ such that $c > s > c - \delta$ implies $s \in O_2$. However, by the definition of the supremum we find $s \in O_1$ between $c - \delta$ and c. This shows that $O_1 \cap O_2$ is not empty. ■

The following lemma follows immediately from the definition.

LEMMA 7.12. *Let $f : X \to Y$ be a continuous function and $E \subset X$ a connected set. Then $f(E)$ is connected.*

COROLLARY 7.13 (Mean-value theorem for continuous functions). *Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be continuous. Then $f(I)$ is an interval.*

8. Picard-iteration

THEOREM 8.1. *Let $\emptyset \neq X$ be a complete metric space and $T : X \to X$ Lipschitz map with constant less than 1. Then T has a unique fixpoint.*

PROOF. Since $T : X \to X$ has Lipschitz map with constant $c < 1$ we known that

$$d(T(x), T(y)) \leq cd(x, y)$$

holds for all $x, y \in X$. We define inductively $T^0 = id$ and $T^{n+1} = T \circ T^n$. Let $x_0 \in X$. We consider the sequence $(x_n)_{n \geq 0}$ defined by $x_n = T^n(x)$. Note that

$$d(x_{n+1}, x_n) \;=\; d(T(x_n), T(x_{n-1}) \;\leq\; cd(x_n, x_1) \;\leq\; \cdots \;\leq\; c^n d(x_1, d_0)\,.$$

Using the geometric series, we deduce

$$\begin{aligned} d(x_{n+m}, x_n) \leq \sum_{k=1}^{m-1} d(x_{n+k}, x_{n+k-1}) \;&\leq\; \sum_{k=1}^{m-1} c^{n+k-1} d(x_1, x_0) \\ &\leq c^n d(x_1, x_0) \sum_{k=0}^{\infty} c^k \;=\; c^n d(x_1, x_0) \frac{1}{1-c}\,. \end{aligned}$$

Thus (x_n) is Cauchy. Since X is complete $x = \lim_n x_n$ exists. Then we note that the continuity of $d : X \times X \to \mathbb{R}$ implies

$$d(T(x), x) \;=\; d(\lim_n T(x_n), \lim_n x_{n+1}) \;=\; \lim_n d(x_{n+1}, x_{n+1}) \;=\; 0\,.$$

We deduce $T(x) = x$. If x' is another fixpoint we we have

$$d(x, x') \;=\; d(T(x), T(x')) \;\leq\; cd(x, x')$$

Thus $(1-c)d(x, x') \leq 0$ implies $d(x, x') = 0$ and hence $x = x'$. ■

EXAMPLE 8.2. *The function $f(x) = 1 - x$ has a unique fixpoint. However, the iterates $x_n = f^n(x)$ only converge for $x = \frac{1}{2}$.*

THEOREM 8.3. *Let $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous and $L > 0$ such that*

$$|\phi(s,t) - \phi(s,r)| \ \le \ L|t-r| \, .$$

Let $y_0 \in \mathbb{R}$ such that $|\phi(s-x_0, y_0)| \ \le \ Ce^{C|s|}$ for some constant $C > 0$. Then there exists a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) \ = \ \phi(x, f(x)) \quad \textit{and} \quad f(x_0) \ = \ y_0 \, .$$

PROOF. We will show the assertion for $x_0 = 0$. Let $\alpha > \max(L, C)$. We define X_α to be the set of continuous functions such that $\sup_{t\in\mathbb{R}} e^{-\alpha|t|}|f(t) - y_0| < \infty$. We define the distance function

$$d(f,g) \ = \ \sup_t e^{-\alpha|t|}|f(t) - g(t)| \, .$$

Note that if $f \in X_\alpha$ and $d(f,g) < \infty$, then $g \in X_\alpha$. It is easily checked that X_α is a complete metric space (see section 3 for a similar argument). We define

$$T(f)(t) \ = \ y_0 + \int_0^t \phi(s, f(s))ds \, .$$

Since ϕ and f are continuous the function $g(s) \ = \ \phi(s, f(s))$ is also continuous. In particular, $T(f)$ is a continuous function. The key calculation here is the Lipschitz constant of T: Let f and g be continuous functions such that $d(f,g) < \infty$. Then we have

$$\begin{aligned}
|T(f)(t) - T(g)(t)| &= \left| \int_0^t (\phi(s, f(s)) - \phi(s, g(s)))ds \right| \\
&\le \int_0^t |\phi(s, f(s)) - \phi(s, g(s))|ds \\
&\le L \int_0^t |f(s) - g(s))|ds \\
&\le L \int_0^t e^{\alpha|s|} ds d(f,g) \\
&\le \frac{L}{\alpha}(e^{\alpha|s|} - 1)d(f,g) \ \le \ \frac{L}{\alpha} e^{\alpha|s|} d(f,g) \, .
\end{aligned}$$

We define $c \ = \ L/\alpha < 1$ and deduce that

$$d(T(f), T(g)) \ \le \ cd(f,g) \, .$$

Now we consider $f_0(t) = y_0$. Then, we have

$$d(T(f), f_0) \;\le\; d(T(f), T(f_0)) + d(T(f_0), f_0) \;\le\; cd(f, f_0) + d(T(f_0), f_0)\ .$$

We observe that $\alpha > C$ implies

$$d(T(f_0), f_0) \;\le\; \sup_t e^{-\alpha|t|} \left| \int_0^t \phi(s, y_0) ds \right| \;\le\; C \sup_t e^{-\alpha|t|} \frac{e^{C|t|} - 1}{C} \;\le\; 1\ .$$

Therefore $T(X_\alpha) \subset X_\alpha$. By the contraction principle we find a unique $f \in X_\alpha$ such that $T(f) = f$. This means

$$f(t) \;=\; y_0 + \int_0^t \phi(s, f(s)) ds\ .$$

In particular, $f(0) = y_0$. The fundamental theorem of calculus implies $f'(t) = \phi(t, f(t))$. ■

We consider $\phi(s, r) = r$ and $y_0 = 1$. Then the iterations are given by

$$f_0(t) = 1\ ,$$

$$f_1(t) = 1 + \int_0^t ds \;=\; 1 + t$$

$$f_2(t) = 1 + \int_0^t (1 + s) ds \;=\; 1 + t + \frac{t^2}{2}$$

$$f_3(t) = 1 + \int_0^t (1 + s + \frac{s^2}{2}) ds \;=\; 1 + t + \frac{t^2}{2} + \frac{t^3}{3!}$$

We obtain the Taylor series

$$f_n(t) \;=\; \sum_{k=0}^{n} \frac{t^k}{k!}$$

By our theorem we know that for every $\alpha > 1$

$$\lim_n \sup_{t \ge 0} e^{-\alpha t} |f_n(t) - e^t| \;=\; 0\ .$$

This is not true for $\alpha = 1$.

What happens if we want to solve

$$f'(t) \;=\; f(t)^2\ ?$$

The function $\phi(r) = r^2$ is not Lipschitz on $\mathbb{R}$. We have to use a local variant.

THEOREM 8.4. *Let I and J be compact intervals with midpoints x_0 and y_0. Let $\phi : I \times J \to \mathbb{R}$ be a continuous function such that*

$$|\phi(s,r) - \phi(s,t)| \;\leq\; L|r-t|$$

for some constant L. Then there exists a $h > 0$ such that

$$f'(x) \;=\; \phi(x, f(x)) \quad , \quad f(x_0) = y_0$$

has a unique solution on $(x_0 - h, x_0 + h)$.

PROOF. We assume again $x_0 = 0$, $I = [-a, a]$ and $J = [y_0 - b, y_0 + b]$. We choose $\alpha > L$. Let $h > 0$ such that $[x_0 - h, x_0 + h] \subset I$. We define the subset C of $X_\alpha[x_0 - h, x_0 + h]$ as

$$C \;=\; \{f \;:\; f(I) \subset J\} \;.$$

Again T is defined as

$$T(f)(t) \;=\; y_0 + \int_0^t \phi(s, f(s))ds \;.$$

Let $f \in C$. Then we have

$$\begin{aligned}
|T(f)(t) - y_0| &= \left| \int_0^t \phi(s, f(s))ds \right| \\
&\leq \left| \int_0^t (\phi(s, f(s)) - \phi(s, y_0))ds \right| + \left| \int_0^t \phi(s, y_0)ds \right| \\
&\leq \left| \int_0^t L|f(s) - y_0|ds \right| + |t| \sup_{s \in I} |\phi(s, y_0)| \\
&\leq h(Lb + \sup_{s \in I} |\phi(s, y_0)|) \;.
\end{aligned}$$

Hence for $h \leq \frac{b}{Lb + \sup_{s\in I} |\phi(s,y_0)|}$ we have $T(C) \subset C$. Since C is a closed subset of $X_\alpha[-h, h]$ we may apply the contraction principle. ∎

For the example $f'(x) = f(x)^2$ with $f(0) = 1$ we find

$$f_0(t) = 1 \;,$$

$$f_1(t) = 1 + \int_0^t ds \;=\; 1 + t$$

$$f_2(t) = 1 + \int_0^t (1+s)^2 ds \;=\; 1 + \int_0^t (1 + 2s + s^2) ds \;=\; 1 + t + t^2 + \frac{t^3}{3}$$

$$f_3(t) = 1 + \int_0^t (1 + s + s^2 + \frac{s^3}{3})^2 ds \;=$$

$$= 1 + \int_0^t (1 + s^2 + s^4 + \frac{s^6}{9} + 2s + 2s^2 + 2\frac{s^3}{3} + \cdots) ds$$

$$= 1 + t + t^2 + t^3 + \cdots$$

The solution is $f(t) = \sum_{k=0}^{\infty} t^k \;=\; \frac{1}{1-t}$. Of course this can not be extended to $\mathbb{R}$ and uniform convergence only works on compact subsets of $(-1, 1)$.