Take-home exam

Name:

(1) (20P) Find the definition of the Zariski topology on $\mathbb{C}^n$ and find a function $f : \mathbb{C}^n \to \mathbb{C}$ which is continuous with respect to the ordinary metric $d(x,y) = (\sum_k |x_k - y_k|^2)^{1/2}$ but not continuous with respect to the Zariski topology.

(2) (40P) Use the results from the lecture and show that for every continuous function $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying

$$|\phi(s,t) - \phi(s,r)| \le L|t-r|$$

and every $(x_0, y_0) \in \mathbb{R}^2$ there exists one and only one solution for

$$f'(x) \;=\; \phi(x, f(x)) \quad f(x_0) \;=\; y_0 \,. \tag{0.1}$$

(Hint: Using the Lipschitz condition you can show that there exist a unique continuous function $f : [x_0 - 2, x_0 + 2] \to \mathbb{R}$, differentiable in $(x_0 - 2, x_0 + 2)$ such that (0.1) is satisfied. Describe exactly the set where you apply the contraction principle. For the uniqueness investigate two solutions and study points x_0 such that for all $x_0 - \varepsilon < x \le x_0$ we have $f(x) = g(x)$.)

(3) (20P) Let D be a dense subset of a compact metric space K . Let $f : D \to \mathbb{R}$ be a function. Sow that f is uniform continuous if and only of there exists a continuous function $F : K \to \mathbb{R}$ such that $F(x) = f(x)$ holds for all $x \in D$.

(4) (32P) Recall the space

$$\ell_1 \;=\; \{(x_n) : \sum_n |x_n| < \infty\}$$

which is equipped with the distance

$$d_1(x,y) \;=\; \sum_n |x_n - y_n| \,.$$

Let (α_n) be a sequence of positive numbers such that $\alpha_n \ge 1$ and define

$$C \;=\; \{(x_n) \;:\; \sum_n |x_n|\alpha_n \le 1\}$$

(a) Show that C is closed.

(b) Now (and in the following) we assume $\lim_n \alpha_n \;=\; \infty$. Show that for every $\varepsilon > 0$ there exists an n_0 such that $(x_n) \in C$ implies

$$\sum_{n \ge n_0} |x_n| < \varepsilon \,.$$

(c) Show that C is totally bounded.

(Hint: You might find it helpful to use that

$$\{(x_n)_{n \le n_0} \; : \; \sum_{n \le n_0} a_n |x_n| \le 1\} \subset \mathbb{R}^{n_0}$$

is compact.)

(d) Assume $\lim_n \alpha_n = \infty$. Show that

$$C = \{(x_n) : \sum_n \alpha_n |x_n| \le 1\}$$

is compact as a subset of ℓ_p for every $1 \le p \le \infty$.