1. Linear differential equations

In the vector-valued situation, Picard’s iteration works equally well.

**Proposition 1.1.** Let $I$ be an interval and $X$ be a Banach space. Let $x_0 \in I$ and $y_0 \in X$ and $\delta > 0$. Let $\phi : I \times \bar{B}(y_0, \delta) \to X$ be a continuous function such that
\[ \| \phi(t, x) - \phi(t, y) \| \leq L \| x - y \| . \]
Then there exists an $h$ such that $f'(t) = \phi(t, f(t))$ with $f(x_0) = y_0$ has a unique solution among continuous functions $f : (x_0 - h, x_0 + h) \to \bar{B}(y_0, \delta)$.

**Proof.** Define
\[ T(f)(t) = y_0 + \int_0^t \phi(s, f(s))ds \]
Let $\alpha > L$ and $h > 0$. We consider the Banach space
\[ Y = \{ f : [x_0 - h, x_0 + h] \to X : f \text{ continuous} \} \]
with the norm
\[ \| f \| = \sup_{|s| \leq h} e^{-\alpha|s|} \| f(x_0 + s) \| . \]
We consider the closed subset $C$ defined by
\[ C = \{ f : f([x_0 - h, x_0 + h]) \subset \bar{B}(y_0, \delta) \} . \]
With the appropriate choice of $h$ we find $T(C) \subset C$. The Banach contraction principle yields a fixpoint $T(f) = f$ which satisfies
\[ f'(t) = \phi(t, f(t)) . \]
For any solution $g'(t) = \phi(t, g(t))$ with $g(x_0) = y_0$ we have
\[ y_0 + \int_{x_0}^t \phi(s, g(s))ds = y_0 + \int_{x_0}^t g'(s)ds = g(t) . \]
Thus we have uniqueness for continuous $g$ with values in $\bar{B}(y_0, \delta)$.

Let us recall the usual trick for transforming DE with linear coefficients into systems:
We are given the differential equation
\[ y''(x) + \sum_{k=0}^{n-1} a_k y^k(x) = 0 \]
with conditions \( y(x_0) = y_0, \ y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}. \) Then we introduce the new variables

\[
y_0 = y, \ y_1 = y', \ldots, \ y_{n-1} = y^{(n-1)}.
\]

This leads to the matrix valued equation

\[
\vec{y}' = A(\vec{y}), \ \vec{y}(x_0) = (y_0, \ldots, y_{n-1})
\]

where

\[
A = \begin{pmatrix}
0 & 1 & 0 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
\vdots \\
0 & 0 & \cdots & 0 & 1 \\
-a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1}
\end{pmatrix}.
\]

**Proposition 1.2.** Let \( X \) be a Banach space and \( A : X \to X \) be linear map. The differential equation

\[
f'(t) = A(t), \quad f(0) = y_0
\]

has the (unique) solution

\[
f(t) = e^{tA}y_0.
\]

**Proof.** We consider the power series

\[
g(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k
\]

with values in \( L(X, X) \). By Lemma 7.10, we deduce

\[
g'(t) = \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!}A^k = Ag(t).
\]

Therefore the chain rule implies that

\[
f(t) = g(t)y_0 = e_{y_0}(g(t))
\]

satisfies

\[
f'(t) = e_{y_0}(g'(t)) = g'(t)y_0 = A(g(t)y_0) = A(f(t)).
\]

The condition \( f(0) = y_0 \) is obvious. (Note that uniqueness follows from the Picard iteration method.)
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LEMMA 1.3. Let $A$ be a complex $n \times n$ matrix and $U$ invertible such that $A = UBU^{-1}$. Then for every power series $f(t) = \sum_k a_k t^k$ converging on $\mathbb{R}$ we have

$$f(A) = U f(B) U^{-1}.$$ 

PROOF. Since power series are uniformly converging it suffice to show the assertion for polynomials. By linearity we only have to consider $f(t) = t^k$. However,

$$A^k = [UBU^{-1}]^k = UBU^{-1}UB\cdots U^{-1}UBU^{-1} = UB^kU^{-1}. \quad \Box$$

LEMMA 1.4. Let $B$ an $n \times n$-Jordan block for the eigenvalue $\lambda$. Then

$$e^{tB} = e^{\lambda t} \left( \sum_{k=0}^n \frac{t^k}{k!} \sum_{j=1}^{n-1-k} e_{j,j+k+1} \right)$$

where $e_{ij}$ are the matrix units.

PROOF. We may write $B = \lambda 1 + C$ where

$$C = \sum_{j=1}^{n-1} e_{j,j+1}$$

is a sum of matrices with only one entry. Note that $e_{r,s}e_{st} = \delta_{rs}e_{tt}$. Then we get

$$C^m = \left( \sum_{j=1}^{n-1} e_{j,j+1} \right)^m$$

$$= \sum_{j_1,\ldots,j_m=1}^{n-1} e_{j_1,j_1+1} \cdots e_{j_m,j_m+1}$$

$$= \sum_{j=1}^{m} e_{j,j+1} e_{j+1,j+2} \cdots e_{j+m,j+m+1}$$

$$= \sum_{j=1}^{m} e_{j,j+m+1}.$$ 

In particular, $C^n = 0$. This yields

$$e^{tC} = \sum_{k=0}^{\infty} \frac{t^k C^k}{k!} = \sum_{k=0}^{n} \frac{t^k}{k!} \sum_{j=1}^{n-1-k} e_{j,j+k+1}.$$ 

Finally we note that that for commuting matrices $e^{A+B} = e^A e^B$ follows from the properties of the binomial coefficients (as for scalars). \quad \Box
An example will illustrate the procedure

\[ C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \]

Then, we have

\[ C^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

Since \( C^3 = 0 \), we get

\[ e^{tC} = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}. \]

Thus the monomials \( \frac{t^n}{n!} \) ‘move’ in the shifted diagonal. Note that the solutions of \( e^{\lambda \gamma} \) should appear as linear combinations of the columns.

**Corollary 1.5.** Let \( A \) be (real) \( n \times n \) matrix. Then the space of solutions

\[ y' = Ay \]

has dimension \( n \). The complex solutions are of the form \( y(t) = e^{\lambda_j t}v(t) \) where \( v \) is a vector of polynomials of degree \( < s_j \) where \( s_j \) is the power of \( \lambda_j \) occurring in the minimal polynomial of \( A \). The function \( g(t) = \det(e^{tA}) \) is called the Wronskian and satisfies

\[ g'(t) = \tr(A)g(t). \]

**Proof.** Every solution is uniquely determined by the initial condition \( y(0) \) and \( y(t) = e^{At}y(0) \). By changing the initial conditions, we may assume that \( A \) is in Jordan normal form. Then the assertion on the particular form follows immediately from the considerations above. The second assertion follows from the chain rule. Indeed for \( t = 0 \) we have

\[ g'(0) = (\det)'(1)(A) = \tr(A). \]

For arbitrary \( t \) we have

\[ \lim_{h \to 0} \frac{\det(e^{t+h}A) - \det(e^{tA})}{h} = \lim_{h \to 0} \frac{\det(e^{hA}) - 1}{h} \det(e^{tA}) = \tr(A)g(t). \]
Lemma 1.6. Let $A$ be the matrix from (8.2). Then

$$\det(A - \lambda) = (-1)^n \left( \sum_{k=0}^{n-1} a_k \lambda^k + \lambda^n \right)$$

Proof. For $n = 2$ we have

$$\det(A - \lambda) = -\lambda(-(a_1 + \lambda)) - (-a_0) = \lambda^2 a_2 + \lambda a_1 + a_0 .$$

Now, we proceed by induction and develop the determinant after the first column. Then

$$\det(A - \lambda) = -\lambda \det_{n-1}((A - \lambda)_{1,1}) + (-1)^{n-1}(-a_0)$$

$$= (-\lambda)(-1)^{n-1} \left( \sum_{k=0}^{n-2} a_{k+1} \lambda^k + \lambda^{n-1} \right) + (-1)^n a_0$$

$$= (-1)^n \left( \sum_{k=0}^{n-1} a_k \lambda^k + \lambda^n \right).$$

Proposition 1.7. Let $p(x) = \sum_{k=0}^{n} a_k x^k$ be a polynomial of degree $n$ with $a_n = 1$. Let $\lambda_1, \ldots, \lambda_m$ be the roots of the polynomial and

$$p(x) = \prod_{j=1}^{m} (x - \lambda_j)^{r_j} .$$

Then the system

$$e^{\lambda_1 t}, \ldots, e^{\lambda_1 t r_1 - 1}, \ldots, e^{\lambda_m t}, \ldots, e^{\lambda_m t r_m - 1}$$

is basis for the space of solutions of

$$\sum_{k=0}^{n} a_k y^{(k)} = 0 .$$

Proof. On the algebra $C_\infty(\mathbb{R}, \mathbb{C})$ of infinitely often differentiable maps we define the linear map

$$D(f) = f' .$$

The our problem is to find solutions for

$$p(D)f = 0 .$$

Note that

$$p(x) = \prod_{j=1}^{m} (x - \lambda_j)^{r_j}$$

implies (after doing the math) that

$$p(D) = \prod_{j=1}^{m} (D - \lambda_j)^{r_j} .$$
Now, we consider \( j = 1, \ldots, m \) and \( f_k(t) = e^{\lambda_j t} t^k \). Note that
\[
(D - \lambda_j)f_k = (D - \lambda_j)e^{\lambda_j t} t^k = (e^{\lambda_j t} t^k)' - \lambda_j e^{\lambda_j t} t^k \\
= \lambda_j e^{\lambda_j t} t^k + e^{\lambda_j t} k t^{k-1} - \lambda_j e^{\lambda_j t} t^k = ke^{\lambda_j t} t^{k-1}.
\]
By induction we get
\[
(D - \lambda_j)^r f_k = k(k-1) \cdots (k-r+1) f_{k-r}.
\]
Thus for \( k < r \) we always find 0. Collection all the solutions we have found \( n \) solutions \( y_1, \ldots, y_n \). Using the matrix \( A \) from (8.2) we may write
\[
\vec{y}_j(t) = e^{tA} \vec{y}_j(0)
\]
where
\[
\vec{y}(t) = \begin{pmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{n-1}(t) \end{pmatrix}.
\]
In order to show linear independence we recall that by Lemma 6.6 we know the characteristic polynomial for \( A \) and thus Corollary 6.5 tells us that we find a system of \( n \) linear independent solutions in the span of the solutions discovered above. Thus the solutions have to be linearly independent.