1. Linear differential equations

In the vector-valued situation, Picard's iteration works equally well.

PROPOSITION 1.1. *Let I be an interval and X be a Banach space. Let $x_0 \in I$ and $y_0 \in X$ and $\delta > 0$. Let $\phi : I \times \bar{B}(y_0, \delta) \to X$ be a continuous function such that*

$$\|\phi(t,x) - \phi(t,y)\| \;\le\; L\|x-y\| \,.$$

Then there exists an h such that

$$f'(t) \;=\; \phi(t, f(t)) \quad f(x_0) \;=\; y_0$$

has a unique solution among continuous functions $f : (x_0 - h, x_0 + h) \to B(y_0, \delta)$.

PROOF. Define

$$T(f)(t) \;=\; y_0 + \int_0^t \phi(s, f(s))ds$$

Let $\alpha > L$ and $h > 0$. We consider the Banach space

$$Y \;=\; \{f : [x_0 - h, x_0 + h] \to X \;:\; f \text{ continuous } \}$$

with the norm

$$\|f\| \;=\; \sup_{|s|\le h} e^{-\alpha|s|}\|f(x_0 + s)\| \,.$$

We consider the closed subset C defined by

$$C \;=\; \{f \;:\; f([x_0 - h, x_0 + h]) \subset \bar{B}(y_0, \delta)\} \,.$$

With the appropriate choice of h we find $T(C) \subset C$. The Banach contraction principle yields a fixpoint $T(f) = f$ which satisfies

$$f'(t) \;=\; \phi(t, f(t)) \,.$$

For any solution $g'(t) = \phi(t, g(t))$ with $g(x_0) = y_0$ we have

$$y_0 + \int_{x_0}^t \phi(s, g(s))ds \;=\; y_0 + \int_{x_0}^t g'(s)ds \;=\; g(t) \,.$$

Thus we have uniqueness for continuous g with values in $\bar{B}(y_0, \delta)$. ∎

Let us recall the usual trick for transforming DE with linear coefficients into systems: We are given the differential equation

lde (1.1)
$$y^n(x) + \sum_{k=0}^{n-1} a_k y^k(x) \;=\; 0$$

with conditions $y(x_0) = y_0$, $y'(x_0) = y_1$,....,$y^{(n-1)}(x_0) = y_{n-1}$. Then we introduce the new variables

$$y_0 \;=\; y \;,\; y_1 \;=\; y' \;,\cdots ,\; y_{n-1} \;=\; y^{(n-1)} \;.$$

This leads to the matrix valued equation

$$\vec{y}' \;=\; A(\vec{y}) \quad , \quad \vec{y}(x_0) \;=\; (y_0, ..., y_{n-1})$$

where

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$$A \;=\; \begin{pmatrix} 0 & 1 & 0\cdots & & \\ 0 & 0 & 1 & 0\cdots & \\ \vdots & & & & \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{pmatrix} . \tag{1.2}$$

Proposition 1.2. *Let X be a Banach space and $A : X \to X$ be linear map. The differential equation*

$$f'(t) \;=\; A(t) \quad , \quad f(0) \;=\; y_0$$

has the (unique) solution

$$f(t) \;=\; e^{tA} y_0 \;.$$

Proof. We consider the power series

$$g(t) \;=\; \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

with values in $L(X, X)$. By Lemma pow 7.10, we deduce

$$g'(t) \;=\; \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} A^k \;=\; Ag(t) \;.$$

Therefore the chain rule implies that

$$f(t) \;=\; g(t) y_0 \;=\; e_{y_o}(g(t))$$

satisfies

$$f'(t) \;=\; e_{y_0}(g'(t)) \;=\; g'(t) y_0 \;=\; A(g(t) y_0) \;=\; A(f(t)) \;.$$

The condition $f(0) = y_0$ is obvious. (Note that uniqueness follows from the Picard iteration method.) ∎

LEMMA 1.3. *Let A be a complex $n\times n$ matrix and U invertible such that $A = UBU^{-1}$. Then for every power series $f(t) = \sum_k a_k t^k$ converging on $\mathbb{R}$ we have*

$$f(A) \;=\; Uf(B)U^{-1}\,.$$

PROOF. Since power series are uniformly converging it suffice to show the assertion for polynomials. By linearity we only have to consider $f(t) = t^k$. However,

$$A^k \qquad = [UBU^{-1}]^k = \underbrace{UBU^{-1}UB\cdots U^{-1}UBU^{-1}}_{k\text{-terms}} \;=\; UB^kU^{-1}\,. \qquad \blacksquare$$

LEMMA 1.4. *Let B an $n\times n$-Jordan block for the eigenvalue λ. Then*

$$e^{tB} \;=\; e^{t\lambda}\left(\sum_{k=0}^{n}\frac{t^k}{k!}\sum_{j=1,\dots,n-1-k} e_{j,j+k+1}\right)$$

where e_{ij} are the matrix units.

PROOF. We may write $B \;=\; \lambda 1 + C$ where

$$C \;=\; \sum_{j=1}^{n-1} e_{j,j+1}$$

is a sum of matrices with only one entry. Note that $e_{lr}e_{st} \;=\; \delta_{rs}e_{lt}$. Then we get

$$\begin{aligned}
C^m &= (\sum_{j=1}^{n-1} e_{j,j+1})^m \\
&= \sum_{j_1,\dots,j_m=1}^{n-1} e_{j_1,j_1+1}\cdots e_{j_m,j_m+1} \\
&= \sum_{j=1,\dots,n-1,j+m\le n-1}^{m} e_{j,j+1}e_{j+1,j+2}\cdots e_{j+m,j+m+1} \\
&= \sum_{j=1,\dots,n-1-m}^{m} e_{j,j+m+1}\,.
\end{aligned}$$

In particular, $C^n = 0$. This yields

$$\begin{aligned}
e^{tC} &= \sum_{k=0}^{\infty}\frac{t^kC^k}{k!} \\
&= \sum_{k=0}^{n}\frac{t^k}{k!}\sum_{j=1,\dots,n-1-k} e_{j,j+k+1}\,.
\end{aligned}$$

Finally we note that that for commuting matrices $e^{A+B} \;=\; e^Ae^B$ follows from the properties of the binomial coefficients (as for scalars). $\blacksquare$

An example will illustrate the procedure

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} .$$

Then, we have

$$C^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

Since $C^3 = 0$, we get

$$e^{tC} = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} .$$

Thus the monomials $\frac{t^k}{k!}$ 'move' in the shifted diagonal. Note that the solutions of (lde 8.1) should appear as linear combinations of the columns.

wrons COROLLARY 1.5. *Let A be (real) $n \times n$ matrix. Then the space of solutions*

$$y' = A(y)$$

has dimension n. The complex solutions are of the form $y(t) = e^{\lambda_j t} v(t)$ where v is a vector of polynomials of degree $< s_j$ where s_j is the power of λ_j occurring in the minimal polynomial of A. The function $g(t) = \det(e^{tA})$ is called the Wronskian and satisfies

$$g'(t) = tr(A)g(t) .$$

PROOF. Every solution is uniquely determined by the initial condition $y(0)$ and $y(t) = e^{At}y(0)$. By changing the initial conditions, we may assume that A is in Jordan normal form. Then the assertion on the particular form follows immediately from the considerations above. The second assertion follows from the chain rule. Indeed for $t = 0$ we have

$$g'(0) = (\det)'(1)(A) = tr(A) .$$

For arbitrary t we have

$$\lim_{h\to 0} \frac{\det(e^{t+h}A) - \det(e^{tA})}{h} = \lim_{h\to 0} \frac{\det(e^{hA}) - 1}{h} \det(e^{tA}) = tr(A)g(t) . \qquad \blacksquare$$

dd **Lemma 1.6.** *Let A be the matrix from (maa 8.2). Then*

$$\det(A-\lambda) \;=\; (-1)^n(\sum_{k=0}^{n-1} a_k\lambda^k + \lambda^n)$$

Proof. For $n = 2$ we have

$$\det(A-\lambda) \;=\; -\lambda(-(a_1+\lambda)) - (-a_0) \;=\; \lambda^2 a_2 + \lambda a_1 + a_0 \; .$$

Now, we proceed by induction and develop the determinant after the first column. Then

$$\begin{aligned}\det(A-\lambda) &= -\lambda \det{}_{n-1}((A-\lambda)_{1,1}) + (-1)^{n-1}(-a_0)\\ &= (-\lambda)(-1)^{n-1}(\sum_{k=0}^{n-2} a_{k+1}\lambda^k + \lambda^{n-1}) + (-1)^n a_0\\ &= (-1)^n(\sum_{k=0}^{n-1} a_k\lambda^k + \lambda^n) \; .\end{aligned}$$

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Proposition 1.7. *Let $p(x) = \sum_{k=0}^n a_k x^k$ be a polynomial of degree n with $a_n = 1$. Let $\lambda_1, ..., \lambda_m$ be the roots of the polynomial and*

$$p(x) \;=\; \prod_{j=1}^m (x-\lambda_j)^{r_j} \; .$$

Then the system

$$e^{\lambda_1 t}, ..., e^{\lambda_1 t}t^{r_1-1}, \cdots, e^{\lambda_m t}, ..., e^{\lambda_m t}t^{r_m-1}$$

is basis for the space of solutions of

$$\sum_{k=0}^n a_k y^{(k)} \;=\; 0 \; .$$

Proof. On the algebra $C_\infty(\mathbb{R},\mathbb{C})$ of infinitely often differentiable maps we define the linear map

$$D(f) \;=\; f' \; .$$

The our problem is to find solutions for

$$p(D)f \;=\; 0 \; .$$

Note that

$$p(x) \;=\; \prod(x-\lambda_j)^{r_j}$$

implies (after doing the math) that

$$p(D) \;=\; \prod(D-\lambda_j)^{r_j} \; .$$

Now, we consider $j = 1, ..., m$ and $f_k(t) \ = \ e^{\lambda_j t} t^k$. Note that

$$\begin{aligned} (D - \lambda_j) f_k = (D - \lambda_j) e^{\lambda_j t} t^k \ &= \ (e^{\lambda_j t} t^k)' - \lambda_j e^{\lambda_j t} t^k \\ &= \lambda_j e^{\lambda_j} t^k + e^{\lambda_j} k t^{k-1} - \lambda_j e^{\lambda_j t} t^k \ = \ k e^{\lambda_j t} t^{k-1} \, . \end{aligned}$$

By induction we get

$$(D - \lambda_j)^r (f_k) \ = \ k(k-1) \cdots (k-r+1) f_{k-r} \, .$$

Thus for $k < r$ we always find 0. Collection all the solutions we have found n solutions $y_1, ...y_n$. Using the matrix A from (8.2) we may write

$$\vec{y_j}(t) \ = \ e^{tA} \vec{y_j}(0)$$

where

$$\vec{y}(t) \ = \ \begin{pmatrix} y(t) \\ y'(t) \\ \cdots \\ y^{n-1}(t) \end{pmatrix} \, .$$

In order to show linear independence we recall that by Lemma 8.6 we know the characteristic polynomial for A and thus Corollary 8.5 tells is that we find a system of n linear independent solutions in the span of the solutions discovered above. Thus the solutions have to be linearly independent. ■