1. Three proofs

PROPOSITION 1.1. *Let* (f_n) *be the Cauchy sequence constructed for* $b = \frac{1}{2}$*. Then there is no continuous function such that* $d_1 - \lim_n f_n = f$*.*

PROOF. Assume $d_1 - \lim_n f_n = f$ with continuous. We will give three proofs for $f(t) = 1$ for $0 < t < \frac{1}{2}$ and $f(t) = 0$ for $\frac{1}{2} < t < 1$. This is of course impossible for a continuous function f. The start of these three proof is the same. Let us assume that $0 < t < \frac{1}{2}$ and $f(t) \neq 1$. Let us assume $f(t) > 1$ .We define $\varepsilon = \frac{f(t)-1}{2}$. By continuity we may find $\delta > 0$ such that $t + \delta < 1$ and $t - \delta > 0$ and

$$f(s) > 1 + \varepsilon$$

for all $|t - s| \leq \delta$.

Proof 1. We choose $n > n_0$ such that $f_n(s) = 1$ on $I = [t - \delta, t + \delta]$ and

$$d_1(f - f_n, 0) < \frac{\varepsilon\delta}{2}$$

for all $n > n_0$. We apply the Chebychev's inequality for $\lambda = \varepsilon$ and $\gamma = \frac{\varepsilon}{3}$ and find intervals $J_1, ..., J_m$ such that

$$\{s \in [0,1] \ : \ |f(s) - f_n(s)| > \varepsilon\} \subset J_1 \cup \cdots \cup J_m$$

and

$$\sum_k |J_k| \ \leq \ \frac{1}{\lambda - 2\gamma} \int_0^1 |f_n(s) - f(s)| ds \ \leq \ \frac{3}{\varepsilon} d_1(f_n, f) < \ .$$

Therefore the interval I is contained in

$$I \subset J_1 \cup \cdots \cup J_m$$

By renumbering the intervals J_k we may assume that there are $a_k \in [t - \delta, t + \delta]$ such that $a_0 = t - \delta$, $a_{m+1} = t + \delta$ and

$$[a_{k-1}, a_k] \subset J_k$$

This yields

$$2\delta \ = \ |I| \ \leq \ \sum_k |J_k| \ \leq \ \frac{3}{\varepsilon} d_1(f_n, f) < 3\frac{\delta}{3} \ = \ \delta \ .$$

This concludes the second proof. ∎

Proof 2. Modifying the proof of Proposition
reflimit1, we know that for a suitable subsequence f_{n_k} converges to a function f a.e. Therefore $f_{n_k} 1_{[t-\delta,t+\delta]}$ converges a.e. to $f 1_{[t-\delta,t+\delta]}$. This means that there exists a set A of measure 0 such that for all $s \in [t-\delta, t+\delta] \setminus A$ we have

$$f(s) \;=\; \lim_k f_{n_k}(s) \;=\; 1 \,.$$

So it suffices to show that I is not contained in A! Let us assume $I \subset A$. Then we find intervals J_k such

$$A \subset \bigcup_k J_k \quad \text{and} \quad \sum_k |J_k| < \frac{\delta}{3} \,.$$

For every $k \in \mathbb{N}$ and $J_k = [a_k, b_k]$ we define $\varepsilon_k = 2^{-k}\frac{\delta}{6}$ and

$$J_k' \;=\; (a_k - \varepsilon_k, a_k + \varepsilon_k) \,.$$

Then

$$I \subset A \subset \bigcup_k J_k'$$

We will learn in the next section that I is compact and hence there exists $m \in \mathbb{N}$ such that

$$I \subset J_1' \cup \cdots \cup J_m'$$

By Proof 2 we deduce that

$$2\delta \;=\; |I| \;\le\; \sum_{k=1}^{m} |J_k'| \;\le\; \sum_k (|J_k| + 2 2^{-k}\frac{\delta}{6}) \;=\; \sum_k |J_k| + \frac{\delta}{3} \;=\; \frac{2}{3}\delta \,.$$

This contradiction concludes the proof. ■

Here is the easy proof.

Proof 3. We define the map

$$I_\delta(g) \;=\; \int_{t-\delta}^{t+\delta} g(s)ds$$

By the definition of the Riemann integral we know that

$$|I_\delta(g)| \le |\int_{t-\delta}^{t+\delta} g(s) - h(s)ds| \;\le\; \int_0^1 |g(s) - h(s)|ds \;=\; d_1(g,h) \,.$$

Thus we may choose n_0 such that for all $n > n_0$

$$|I_\delta(f - f_n)| \;\le\; \varepsilon\delta$$

Then we may choose also $n > n_0$ such that $f_n(s) = 1$ for all $t - \delta < s < t + \delta$. Then we see that

$$2\varepsilon\delta \leq \int_{t-\delta}^{t+\delta} (f(s) - 1)ds \ \leq \ |I(f - f_n)| < \varepsilon\delta$$

This contradiction concludes the proof. ■

We are done. ■

REMARK 1.2. *We proved and important fact here: If*

$$I \subset \bigcup_k J_k$$

then

$$|I| \ \leq \ \sum_k |J_k| \ .$$

This is the beginning of measure theory. Moreover, we showed that if (f_{n_k}) *converges to* f *a.e. Then for every interval* $I_m = [t - \frac{1}{2}, t]$ *there is an* $s_m \in I_m$ *such that*

$$\lim_k f_{n_k}(s_m) \ = \ f(s_m) \ .$$

Of course this suffices to show that if in our example f *is continuous, then* $f(\frac{1}{2}) = \lim_{s_m \to \frac{1}{2}, s_m < \frac{1}{2}} f(s_m) = 1$ *and* $f(\frac{1}{2}) = \lim_{s'_m \to \frac{1}{2}, s'_m > \frac{1}{2}} = 0$ *for suitable* $s'_m \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{m}]$.