## 1. The implicit and inverse function theorem

In the inverse function theorem we want to solve the equation

impl

$$F(x,y) \;=\; z \tag{1.1}$$

for a function $F : B(x_0,\delta) \times B(y_0,\eta) \to Z$ such that $F(x_0,y_0) = z$. The proof will show that a unique solutions exists (under suitable assumptions) but only in a small neighborhood of $(x_0,y_0)$. Global solutions may not exists for geometrical reasons. For convenience we shall assume $z = 0$. The whole point of the proof is in transforming (impl 1.1) in fixpoint problem.

Remark 1.1. *Let $F : B(x_0,\delta) \times B(y_0,\eta) \to Z$ be differentiable at $(x_0,y_0)$. Then the linear map $F' : X \times Y \to Z$ has two components $F' = (D_1F(x_0,y_0), D_2F(x_0,y_0))$ such that*

$$\lim_{\|x-x_0\|+\|y-y_0\|\to 0} \frac{\|F(x,y) - F(x_0,y_0) - D_1F(x_0,y_0)(x-x_0) + D_2F(x_0,y_0)(y-y_0)\|}{\|x-x_0\| + \|y-y_0\|}$$
$$= \; 0 \,.$$

In the following we will assume that $L_0 = D_2F(x_0,y_0) \to Z$ is a continuous linear map with continuous inverse $L_0^{-1} : Z \to Y$. (For finite dimension this means some matrix has full rank). Then, we deduce that $0 \;=\; F(x,y)$ holds iff

$$0 \;=\; L_0^{-1}F(x,y)$$

iff

$$y \;=\; y - L_0^{-1}F(x,y)\,.$$

Thus we are looking for a function $y = f(x)$ which is the fixpoint for $G(x,y)$.

Lemma 1.2. *Let $Y$ be a Banach space, $D \subset X$ and $C \subset Y$ be a closed bounded, non-empty set. Let $G : D \times C \to C$ be a function such that*

$$\|G(x,y) - G(x,y')\| \;\le\; \alpha\|y - y'\|$$

*holds for some $\alpha < 1$. Then there exists a continuous function $f : D \to C$ such that*

$$G(x,f(x)) \;=\; f(x)$$

*holds for all $x \in D$.*

PROOF. We apply the contraction principle for $Z = C(D, Y)$ and the complete metric space $B = \{f : D \to C\} \subset Z$ with the distance

$$d(f, g) \;=\; \sup_{x\in D} \|f(x) - g(x)\| \,.$$

Then, we define

$$T(f)(x) \;=\; G(x, f(x)) \,.$$

Note that

$$d(T(f), T(g)) \;=\; \sup_{x} \|G(x, f(x)) - G(x, g(x))\| \;\le\; \alpha \sup_{x} \|f(x) - g(x)\| \;=\; \alpha d(f, g) \,.$$

Thus the contraction principle yields a unique continuous $f \in B$ such that $T(f) = f$. ■

COROLLARY 1.3. *Let $Y$ be a Banach space $X$ be a normed space, $C \subset X$ with $y_0 \in D$. Let $x_0 \in C$ such that $D_2F$ is continuous and $L_0 = D_2F(x_0, y_0)$ is invertible. Let $\alpha > 0$. Then there exits a $\eta > 0$ and $\gamma > 0$ such that $G(x, y) = y - L_0^{-1}F(x, y)$ satisfies*

$$\|G(x, y_1) - G(x, y_2)\| \;\le\; \alpha\|y_1 - y_2\|$$

*for all $\|x - x_0\| \le \gamma$, $\|y_1 - y_0\| \le \eta$ and $\|y - y_2\| \le \eta$.*

PROOF. We use the fundamental theorem and get

$$\begin{aligned}
\|G(x, y_1) - G(x, y_2)\| &= \| \int_0^1 D_2G(x, y_2 + t(y_1 - y_2)(y_1 - y_2)dt\| \\
&\le \int \|D_2G(x, y_2 + t(y_1 - y_2)(y_1 - y_2)\|dt \\
&\le \sup_{y=y_2+t(y_1-y_2)} \|D_2G(x, y_2 + t(y_1 - y_2)\|\|y_1 - y_2\| \,.
\end{aligned}$$

Now, we calculate the derivative of $G$ and find

$$D_2G \;=\; Id - L_0^{-1}DF_2(x, y) \;=\; L_0^{-1}(D_2F(x_0, y_0) - F_2(x, y)) \,.$$

By continuity we find $\gamma$, $\eta$ such that $\|x - x_0\| \le \gamma$ and $\|y - y_0\| \le \eta$ implies

$$\|D_2F(x_0, y_0) - D_2F(x, y)\| \;\le\; \frac{\alpha}{\|L_0^{-1}\|} \,.$$

This yields the Lipschitz estimate. However, we also have to check that for $C = = \{y \,:\, \|y - y_0\| \le \eta\}$ we have $G : D \times C \to C$ for a suitable $D = B(x_0, \gamma')$. Indeed, we may find $0 < \gamma' \leq \gamma$ such that $\|x - x_0\| \leq \gamma'$ implies $\|G(x, y_0) - G(x_0, y_0)\| \leq \eta(1-\alpha)$. Then we deduce from $\|y - y_0\| \leq \eta$ that

$$\begin{aligned}\|G(x,y) - y_0\| &\leq \|G(x,y) - G(x,y_0)\| + \|G(x,y_0) - G(x_0,y_0)\| \\ &\leq \alpha\|y - y_0\| + (1-\alpha)\eta \ \leq \ \eta \,.\end{aligned}$$

For later use note that $G : D \times B(y_0, \eta) \subset B(y_0, \eta)$. ■

Thus assuming continuity of $D_2F$ and $D_2F(x_0, y_0)$ we can find a continuous function $f : \bar{B}(x_0, \gamma) \to \bar{B}(y_0, \eta)$ such that

$$f(x) \ = \ G(x, f(x))$$

and hence

$$0 \ = \ F(x, f(x)) \,.$$

If we can show that $f$ is also differentiable we deduce from the chain rule

$$0 \ = \ D_1F(x, f(x)) + D_2F(x, f(x))f'(x)$$

This yields

deriv (1.2)
$$f'(x) \ = \ -(D_2F(x, f(x)))^{-1} D_1F(x, f(x)) \,.$$

The next Lemma ensures invertibility of $D_2F(x, f(x))$.

LEMMA 1.4. *Let $Y$ be Banach space. The set of (left) invertible map in $L(Y, Z)$ is open.*

PROOF. Let $L_0$ be invertible and assume

$$\|L - L_0\| \ \leq \ \frac{\alpha}{\|L_0\|} \,.$$

We consider $T = L_0^{-1}L$ and the series

$$R \ = \ \sum_n T^n$$

which satisfies $\sum_n \|T^n\| \ \leq \ \sum_n \alpha^n \ = \ \frac{1}{1-\alpha}$. Then we get

$$R(1-T) \ = \ \sum_n T^n - \sum_{n>0} T^n \ = \ T^0 \ = \ Id \,.$$

This yields $R = (1-T)^{-1}$ and $RL_0^{-1}L \ = \ I$. ■

The proof of (1.2) goes backwords (meaning we have a candidate for $f'$ and have to show it works). This is a little tricky. We will use an extra assumption

$$F \text{ is differenentiable in neighborhood of } (x_0,y_0)\ .$$

We may now assume that $\gamma$ and $\eta$ are chosen such we end up in this neighborhood and that $D_2F(x_1,y_1)$ is invertible whenever $\|x_1-x_0\|\leq \gamma$ and $\|y_1-y_0\|\leq \eta$. Let us now fix $y_1=f(x_1)$ with this property. Let $\varepsilon$ be the error function such that

$$\lim_{\|x-x_1\|+\|y-y_1\|} \frac{\|\varepsilon(x,y)\|}{\|x-x_1\|+\|y-y_1\|} \;=\; 0$$

and for $K=D_1F(x_1,y_1)$ and $L=D_2F(x_1,y_1)$ we have

$$F(x,y) \;=\; F(x_1,y_1)+K(x-x_1)+L(y-y_1)+\varepsilon(x,y) \;=\; K(x-x_1)+L(y-y_1)+\varepsilon(x,y)\ .$$

For $y=f(x)$ we deduce an additional formula for $f(x)$

form (1.3) $$0 \;=\; L^{-1}(0) \;=\; L^{-1}K(x-x_1)+f(x)-f(x_1)+L^{-1}\varepsilon(x,f(x))\ .$$

Since $f$ is continuous, we may choose $\rho>0$ such that $\|x-x_1\|<\rho$ implies

$$\|\varepsilon(x,f(x))\| \;\leq\; (2\|L^{-1}\|)^{-1}(\|x-x_1\|+\|f(x)-f(x_1)\|)\ .$$

Combining this two estimates we obtain

$$\begin{aligned}\|f(x)-f(x_1)\| &\leq \|L^{-1}\|\|K\|\|x-x_1\|+\|L^{-1}\|\|\varepsilon(x,f(x))\|\\ &\leq \|L^{-1}\|\|K\|\|x-x_1\|+\frac{1}{2}\|x-x_1\|+\frac{1}{2}\|f(x)-f(x_1)\|\ .\end{aligned}$$

This yields (this is the ingenious trick)

$$\|f(x)-f(x_1)\| \;\leq\; (2\|L^{-1}\|\|K\|+1)\|x-x_1\|\ .$$

Now, we use (1.3) again and get

$$f(x)-f(x_1)+L^{-1}K(x-x_1) = -L^{-1}\varepsilon(x,f(x))$$

and

$$\begin{aligned}&\lim_{\|x-x_1\|\to 0} \frac{\|-L^{-1}\varepsilon(x,f(x))\|}{\|x-x_1\|}\\ &= \lim_{\|x-x_1\|\to 0} \frac{\|x-x_1\|+\|f(x)-f(x_1)\|}{\|x-x_1\|}\,\frac{\|-L^{-1}\varepsilon(x,f(x))\|}{\|x-x_1\|+\|f(x)-f(x_1)\|}\\ &\leq (2\|L^{-1}\|\|K\|+1)\|L^{-1}\| \lim_{\|x-x_1\|+\|y-y_1\|} \frac{\|\varepsilon(x,y)\|}{\|x-x_1\|+\|y-y_1\|}\\ &= 0\ .\end{aligned}$$

THEOREM 1.5. *(Implicit function theorem) Let* $F : B(x_0, \delta_0) \times B(y_0, \delta_1) \to Z$ *such that* $F$ *is differentiable,* $D_2F$ *is continuously differentiable and* $D_2F(x_0, y_0)$ *has an inverse. Then there exists* $\gamma > 0$ *and* $\eta > 0$ *such that*

$$F(x, y) \;=\; F(x_0, y_0)$$

*has a unique solution* $y = f(x)$ *in* $\bar{B}(x_0, \gamma) \times \bar{B}(y_0, \eta)$*. Moreover,*

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$$\{(x, f(x)) \;:\; \|f(x) - y_0\| < \eta\} \;=\; \{(x, y) \;:\; \|x - x_0\| < \delta, \|y - y_0\| < \eta\} \,. \tag{1.4}$$

*This unique solution is continuously differentiable and satisfies*

$$f'(x) \;=\; -D_2F(x, f(x))^{-1} D_1F(x, f(x)) \,.$$

COROLLARY 1.6. *Let* $X$*,* $Y$ *be Banach spaces,* $\Omega \subset Y$ *open and* $f : \Omega \to X$ *continuously differentiable. In addition, assume that* $f'(y_0) : Y \to X$ *is invertible. Then three exists an open neighborhood* $W$ *of* $y_0$ *and such that* $f(W)$ *is an open neighborhood of* $f(y_0)$ *and a differentiable function* $g : f(V) \to V$ *such that* $gf(y) = y$ *for all* $y \in V$ *and*

$$g'(f(y_0)) \;=\; (f'(y_0))^{-1} \,.$$

PROOF. We consider $F(x, y) \;=\; -x + f(y)$. Then $D_2F = f'$ and invertible by assumption. We find a function $g$ such that

$$0 \;=\; F(x, g(x)) \;=\; -x + f(g(x))$$

defined an a small neighborhood $B(y_0, \delta)$ of $x_0 = f(y_0)$. We deduce from (1.4) that

$$\begin{aligned} &\{(x, g(x)) \;:\; \|x - x_0\| < \delta, \|g(x) - y_0\| < \eta\} \\ &= \{(x, y) \;:\; x \;=\; f(y), \|x - x_0\| < \delta, \|y - y_0\| < \eta\} \,. \end{aligned}$$

This implies that

$$W \;=\; f^{-1}(B(x_0, \delta)) \cap B(y_0, \eta) \;=\; g(B(x_0, \delta)) \cap B(y_0, \eta)$$

is an open subset of $B(y_0, \eta)$. Moreover, we have

$$g'(y) \;=\; -f'(y)^{-1} D_1F \;=\; f'(y)^{-1} \,.$$

Note that $f(g(x)) = x$ implies that $g$ is injective on $B(x_0, \delta)$. Then the restriction of $f$ to $W = g(V)$ is also injective. We also have $V = f(W) = g^{-1}(B(y_0, \eta))$ is open neighborhood of $x_0$ and $g : V \to W$ is bijective with inverse $f$. ■