Check up -problems

(1) Show that a bounded monotone sequence of real numbers is converging.

(2) Calculate the eigenvalues of \[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}.
\]

(3) Formulate an important result from your undergraduate analysis course.

(4) Formulate an important result from your undergraduate linear algebra course.
Due date: Wednesday, September 8

(1) Show that a bounded, monotone sequence is convergent.

(2) Give an \( \varepsilon-\delta \) proof for
\[
\lim_{n \to \infty} \frac{|5n + n^2|}{|n^2 - 5n|} = 1.
\]

(3) Let \( x > 0 \) show that
\[
\lim_{n \to \infty} x^{\frac{1}{n}} = 1.
\]
(Hint: you may use the continuous function \( f(x) = \ln x \) or Bernoulli’s inequality.)

(4) Show that the vectors \( x_1, \ldots, x_n \in \mathbb{R}^n \) given by
\[
x_j = (1, 1, \ldots, 1_{j\text{-th position}}, 0, 0, \ldots, 0)
\]
are linearly independent.

(5) Show that \( A = \{ x \in \mathbb{R} : \exists k \in \mathbb{N} 2k \leq x \leq 2k + 1 \} \) is closed.

(6) Use the \( \varepsilon-\delta \) criterion to show that \( f : [0, \infty) \to [0, \infty) \) given by \( f(x) = \sqrt{x} \) is continuous.
Transition course -hw1

**Due date:** Wednesday, September 8

(1) Show that a bounded, monotone sequence is convergent.

(2) Give an $\varepsilon$-$\delta$ proof for

$$
\lim_{n} \left| \frac{5n + n^2}{n^2 - 5n} \right| = 1.
$$

(3) Let $x > 0$ show that

$$
\lim_{n} x^{\frac{1}{n}} = 1.
$$

(Hint: you may use the continuous function $f(x) = \ln x$ or Bernoulli’s inequality.)

(4) Show that the vectors $x_1, ..., x_n \in \mathbb{R}^n$ given by

$$
x_j = (1, 1, ..., 1, 0, 0, ..., 0)
$$

are linearly independent.

(5) Show that $A = \{x \in \mathbb{R} : \exists k \in \mathbb{N} \ 2k \leq x \leq 2k + 1\}$ is closed.

(6) Use the $\varepsilon$-$\delta$ criterion to show that $f : [0, \infty) \to [0, \infty)$ given by $f(x) = \sqrt{x}$ is continuous.
(1) Let $1 \leq p < \infty$. Show that $\ell_p = \{(x_n) : \left(\sum_n |x_n|^p\right)^{\frac{1}{p}}\}$ is a complete metric spaces with respect to

$$d_p(x, y) = \left(\sum_n |x_n - y_n|^p\right)^{\frac{1}{p}}.$$  

(Hint: Use $d_p(x, y) = \lim_m \left(\sum_{n \leq m} |x_n - y_n|^p\right)^{\frac{1}{p}}$ in the proof of the triangle inequality.)

(2) Use the result from the lecture for a short proof of the fact $\ell_\infty = \{(x_n) : \sup_n |x_n| < \infty\}$ is complete with respect to $d(x, y) = \sup_n |x_n - y_n|$.

(3) Show that $x_n = \sum_{j \leq n} p^j$ is a Cauchy sequence in $(\mathbb{Z}, dd_p)$. Show that there is no limit $x \in \mathbb{Z}$. (Hint use the unique decomposition $x = \sum a_k p^k$, $a_k \in \{0, \ldots, p - 1\}$ for positive integers.)

(4) A function $f : X \to Y$ is uniformly continuous if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in X (d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon).$$

Show that $f : (0, 1) \to \mathbb{R}$, $f(x) = 1/x$ is not uniformly continuous.

(5) The space $C_b(X, \mathbb{R})$ is also a vector space over $\mathbb{R}$. Find $n$-linearly independent elements for

(a) $X = \{1, \ldots, n\}$ with the discrete metric.

(b) $X = [0, 1]$ with the usual metric. (Hint: polynomials?)
Transition-hw3

Due date: September 20.

1. (a) Let \( f : (a, b) \to \mathbb{R} \) be a differentiable function. Show that \( f \) is Lipschitz if and only if
\[
\sup_x |f'(x)|
\]
is finite.

(b) Determine the maximal \( b > 0 \) such that \( f(x) = x^2 - x \) has Lipschitz constant \( \leq 1 \) on \([0, b]\).

2. Let \( X \) be a metric space, \( 0 < c < 1 \) and \( f : X \to X \) such that
\[
d(f(x), f(y)) \leq cd(x, y).
\]
Let \( x_0 \in X \) and define inductively \( x_{n+1} = f(x_n) \). Show that \( (x_n) \) is Cauchy.

Study the function \( f(x) = 1 - x \) on \([0, 1]\) and show that this does not work for \( c = 1 \).

3. Let \( X \) be the completion of \((\mathbb{Z}, dd_p)\) and \( y \in \mathbb{Z} \).

(a) Show that exists a continuous map \( f : X \to X \) such that \( f(x) = x + y \)
for all \( x \in \mathbb{Z} \).

(b) Show that there exists continuous map \( \text{add} : X \times X \to X \) satisfying
\( \text{add}(x, y) = x + y \) for all \( x, y \in \mathbb{Z} \). (Here the distance on \( X \times X \) is given by
\[
d((x_1, x_2), (y_1, y_2)) = d(x_1, y_1) + d(x_2, y_2) \).
\]

(c) What can you say about multiplication? What structure do you expect
for \( X \).

4. (a) Let \( X \) be a metric space and \( f : X \to Y \) be continuous. Show that \( f(K) \) is compact for all \( K \subset X \) compact.

(b) Let \( X \) be compact metric space and \( f : X \to Y \) be bijective continuous map. Show that \( f^{-1} \) is continuous.
Transition course - hw5

Due date: Monday, October 4

(1) On $X = \{-1, 1\}^\mathbb{N} = \{ (\varepsilon_1, \varepsilon_2, \cdots ) : \varepsilon_i = \pm 1 \}$ we define the metric

$$d((\varepsilon_i), (\delta_i)) = \sum_{i=1}^{\infty} 2^{-i} |\varepsilon_i - \delta_i|.$$  

Show that $(X,d)$ is a compact metric spaces. Show that

$$f((\varepsilon_i)) = \sum_i \alpha_i \varepsilon_i$$

is continuous if and only if $\sum_{i} |\alpha_i| < \infty$.

(2) We consider the space $X = C[0,1]$ and

$$F = \{ t \mapsto \sum_{k=0}^{n-1} a_k t^k : \sum_{k=0}^{n-1} |a_k| \leq 1 \}$$

Show that $F$ is compact by finding a compact set $C \subset \mathbb{R}^n$ and a continuous function $g: C \to C[0,1]$ such that $F = f(C)$. 
Transition course -hw5

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$$d((\varepsilon_i), (\delta_i)) = \sum_{i=1}^{\infty} 2^{-i} |\varepsilon_i - \delta_i|.$$ 

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$$F = \{ t \mapsto \sum_{k=0}^{n-1} a_k t^k : \sum_{k=0}^{n-1} |a_k| \leq 1 \}$$

Show that $F$ is compact by finding a compact set $C \subset \mathbb{R}^n$ and a continuous function $g : C \to C[0, 1]$ such that $F = f(C)$. 
Due date: November 1

(1) Calculate the Jordan normal form and appropriate bases of the following matrices.

i) \( A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \)

ii) \( B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \)

(2) Let us recall that \( \det : (F^n)^n \rightarrow F \) is a map which satisfies the following conditions

i) \( \det(v_1 + \lambda w_1, v_2, ..., v_n) = \det(v_1, ..., v_n) + \lambda \det(w_1, ..., v_n) \)

ii) \( \det(v_1, ..., v_{j-1}, v_j, v_{j+1}, v_n) = (-1)^{j+1} \det(v_j, ..., v_{j-1}, v_1, v_{j+1}, v_n) \)

iii) \( \det(e_1, ..., e_n) = 1 \)

for all \( w_1 \in F^n, \lambda \in F, v_1, ..., v_n \in F^n \). Here \( e_1, ..., e_n \) are the standard unit vectors. For a matrix \( A = [a_{ij}] \) we define the column vectors \( v_j = (a_{ij})_{i=1}^n \) and

\[
\det(A) = \det(v_1, ..., v_n).
\]

In the following you may use that there is only one map \( \det : (F^n)^n = \mathcal{M}_n \rightarrow F \) satisfying the conditions i) \( \rightarrow \) iii).

(a) Show that \( \det(AB) = \det(A)\det(B) \).

(b) For a permutation \( \pi : \{1, ..., n\} \), we define a linear map \( T_\pi(e_i) = e_{\pi(i)} \).

Let \( A_\pi \) be the corresponding matrix. We denote the group of permutation by \( S_n \). Show that \( \varepsilon : S_n \rightarrow F \) defined by

\[
\varepsilon(\pi) = \det(A_\pi)
\]

is a group homomorphism.
(c) Let $i \neq j$. Show that for a cycle $(ij)$ (which interchanges $i$ and $j$) we have

$$\varepsilon((ij)) = -1.$$  

(d) Show that every permutation $\pi$ may be written as a product of cycles (hint: use induction) and that for $\pi = (i_1j_1) \cdots (i_mj_m)$ we have

$$\varepsilon(\pi) = (-1)^m.$$  

(One can actually show that every permutation is a product of neighbouring cycles.)
Due date: November 8

(1) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Show that the sequence of polynomials

$$p_n(t) = \sum_{k=0}^{n} \frac{t^k}{k!} A^k$$

converges pointwise in $\mathbb{R}^9$ and calculate the limit. Also find the JNF for $A$.

(2) Do the same as in 1) but now for $A = \begin{pmatrix} 4 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}$ and the series for

$$\cos(t) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!}$$. Also compute the JNF.

(3) A linear map $T : V \to V$ is called nilpotent, if there exists $m \in \mathbb{N}$ such that $T^m = 0$. Let $A$ be an upper diagonal matrix and $T_A$ the induced linear map on $\mathbb{C}^n$. $A$ is called nilpotent if $T_A$ is nilpotent. Characterize nilpotent upper diagonal matrices. Find a nilpotent $2 \times 2$ matrix with non-zero coefficients on the diagonal (hint similarity).

(4) Consider $V = C(\mathbb{R})$, $r > 0$ and $T : C(\mathbb{R}) \to C(\mathbb{R})$ given by $T(f)(t) = f(t + r)$. Calculate $e^T = \sum_{k=0}^{\infty} \frac{T^k}{k!}$ where convergence takes place pointwise. How would you define $T^\frac{1}{2}$ and $e^{iT}$?
Due date: November 8

(1) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Show that the sequence of polynomials $p_n(t) = \sum_{k=0}^{n} \frac{t^k}{k!} A^k$ converges pointwise in $\mathbb{R}^9$ and calculate the limit. Also find the JNF for $A$.

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(4) Consider $V = C(\mathbb{R})$, $r > 0$ and $T : C(\mathbb{R}) \to C(\mathbb{R})$ given by $T(f)(t) = f(t + r)$. Calculate $e^T = \sum_{k=0}^{\infty} \frac{T^k}{k!}$ where convergence takes place pointwise. How would you define $T^\frac{1}{2}$ and $e^{iT}$?
Due date: December 1

(1) Solve the following linear DE
   (a) \( y'' = 2y' - y, \quad y(0) = y'(0) = 1; \)
   (b) \( y'' = 2y' + y, \quad y(0) = y'(0) = 1; \)
   (c) \( y''' = 4y'' - 5y' + y, \quad y(0) = y'(0) = y''(0) = 1 \)
Transition-hw10

**Due date:** December 1

(1) Solve the following linear DE

(a) \( y'' = 2y' - y, \quad y(0) = y'(0) = 1 \);

(b) \( y'' = 2y' + y, \quad y(0) = y'(0) = 1 \);

(c) \( y''' = 4y'' - 5y' + y, \quad y(0) = y'(0) = y''(0) = 1 \)
Due Date: Wednesday, December 8

(1) Let \( F : C[0, 1] \to C[0, 1] \) be the function
\[
F(x)(t) = \int_0^t x(s)ds .
\]
Calculate \( F'(1) \) and show that \( F'(1) \) has no bounded inverse. (Hint: That would imply \( \sup_{0 \leq s \leq 1} |h(s)| \leq c \sup_{t} |\int_0^t h(s)ds| \) for com constant \( c \). However, a clever choice such that \( \int_0^t h(s)ds = \sin(nt) \) makes that pretty impossible).

(2) We want to apply the implicit function theorem for
\[
F(x, y)(t) = \int_0^t x(s)y(s)ds
\]
at \((x_0, y_0) = (1, 1)\). Show that \( D_2 F(1, 1) \) is not invertible and thus the implicit function theorem does not imply. Show that the solution to \( F(x, y) = F(1, 1) \) is given by \( y(s) = \frac{1}{x(s)} \) and that this is well defined for \( \|x - 1\| < 1 \).

(3) A better way to obtain a good solution is to consider the function
\[
F(x, y)(s) = x(s)y(s)
\]
Show that the implicit function theorem applies and yields a map \( u : B(1, \delta) \to C[0, 1] \) such that
\[
F(x, u(x)) = F(1, 1) .
\]
Calculate the derivative.

(4) Show that there exists a differentiable function \( f : [1 - \delta, 1 + \delta] \to \mathbb{R} \) satisfying
\[
(1 + f(x))^{\frac{1}{2}} = f(x)
\]
Calculate the derivative at 1.