(1) Calculate the Jordan normal form and appropriate bases of the following matrices.

i) \( A = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix} \)

ii) \( B = \begin{pmatrix}
2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 3 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3
\end{pmatrix} \)

(2) Let us recall that \( \det : (F^n)^n \to F \) is a map which satisfies the following conditions

i) \( \det(v_1 + \lambda w_1, v_2, ..., v_n) = \det(v_1, ..., v_n) + \lambda \det(w_1, ..., v_n) \)

ii) \( \det(v_1, ..., v_{j-1}, v_j, v_{j+1}, v_n) = (-1)^j \det(v_j, ..., v_{j-1}, v_1, v_{j+1}, v_n) \)

iii) \( \det(e_1, ..., e_n) = 1 \)

for all \( w_1 \in F^n \), \( \lambda \in F \), \( v_1, ..., v_n \in F^n \). Here \( e_1, ..., e_n \) are the standard unit vectors. For a matrix \( A = [a_{ij}] \) we define the column vectors \( v_j = (a_{ij})_{i=1, ..., n} \) and

\( \det(A) = \det(v_1, ..., v_n) \).

In the following you may use that there is only one map \( \det : (F^n)^n = \mathcal{M}_n \to F \) satisfying the conditions i) \( \to iii \).

(a) Show that \( \det(AB) = \det(A)\det(B) \).

(b) For a permutation \( \pi : \{1, ..., n\} \), we define a linear map \( T_\pi(e_i) = e_{\pi(i)} \).

Let \( A_\pi \) be the corresponding matrix. We denote the group of permutation by \( S_n \). Show that \( \varepsilon : S_n \to F \) defined by

\( \varepsilon(\pi) = \det(A_\pi) \)

is a group homomorphism.
(c) Let $i \neq j$. Show that for a cycle $(ij)$ (which interchanges $i$ and $j$) we have
\[ \varepsilon((ij)) = -1. \]

(d) Show that every permutation $\pi$ may be written as a product of cycles (hint: use induction) and that for $\pi = (i_1j_1) \cdots (i_mj_m)$ we have
\[ \varepsilon(\pi) = (-1)^m. \]

(One can actually show that every permutation is a product of neighbouring cycles.)