

Due date: November 1

- (1) Calculate the Jordan normal form and appropriate bases of the following matrices.

$$\text{i) } A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{ii) } B = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

- (2) Let us recall that $\det : (F^n)^n \rightarrow F$ is a map which satisfies the following conditions

- i) $\det(v_1 + \lambda w_1, v_2, \dots, v_n) = \det(v_1, \dots, v_n) + \lambda \det(w_1, \dots, v_n)$
- ii) $\det(v_1, \dots, v_{j-1}, v_j, v_{j+1}, \dots, v_n) = (-1) \det(v_j, \dots, v_{j-1}, v_1, v_{j+1}, \dots, v_n)$,
- iii) $\det(e_1, \dots, e_n) = 1$

for all $w_1 \in F^n$, $\lambda \in F$, $v_1, \dots, v_n \in F^n$. Here e_1, \dots, e_n are the standard unit vectors. For a matrix $A = [a_{ij}]$ we define the column vectors $v_j = (a_{ij})_{i=1, \dots, n}$ and

$$\det(A) = \det(v_1, \dots, v_n).$$

In the following you may use that there is only one map $\det : (F^n)^n = \mathcal{M}_n \rightarrow F$ satisfying the conditions $i) \rightarrow iii)$.

- (a) Show that $\det(AB) = \det(A) \det(B)$.
- (b) For a permutation $\pi : \{1, \dots, n\}$, we define a linear map $T_\pi(e_i) = e_{\pi(i)}$. Let A_π be the cooresponding matrix. We denote the group of permutation by S_n . Show that $\varepsilon : S_n \rightarrow F$ defined by

$$\varepsilon(\pi) = \det(A_\pi)$$

is a group homomorphism.

- (c) Let $i \neq j$. Show that for a cycle (ij) (which interchanges i and j) we have

$$\varepsilon((ij)) = -1.$$

- (d) Show that every permutation π may be written as a product of cycles (hint: use induction) and that for $\pi = (i_1 j_1) \cdots (i_m j_m)$ we have

$$\varepsilon(\pi) = (-1)^m.$$

(One can actually show that every permutation is a product of neighbouring cycles.)