Due date: September 27.

(1) Show the discrete Chebychev inequality: Let \((a_n)\) be positive numbers such that

\[ S = \sum_{n} a_n < \infty \]

is finite. Let \(\lambda > 0\). Show that the set \(A_\lambda = \{ n \in \mathbb{N} : a_n \geq \lambda \}\) has not more than \(S/\lambda\) many elements.

(2) Let \(A_1, \ldots, A_k\) be subset of \(\{1, \ldots, n\}\) such that

\[ \sum_{k} |A_k| < n \]

Show there is an element in \(\bigcup_{j=1}^{k} A^c_k\). (\(A^c_k = \{1, \ldots, n\} \setminus A_k\) is the complement.)

(3) Show that the countable union of sets of measure 0 has measure 0.

(4) We want to show that a bounded function is Riemann integrable if and only if the set of discontinuity points has measure 0. Let \(f : [0, 1] \to \mathbb{R}\) be a function such that

\[ \sup_{x \in [0, 1]} |f(x)| \leq B. \]

We define

\[ \omega_\delta(x) = \sup\{|f(y) - f(x)| : |x - y| < \delta, |x - z| < \delta\}. \]

(a) Show that \(\omega_\delta(x) \leq 2B\).

(b) Let \(I = [a, b]\) be an interval and \(a + \delta \leq x \leq b - \delta\). Show that

\[ \omega_\delta(x)(b - a) \leq (\sup_{I} f - \inf_{I} f)(b - a). \]

(c) Let \(N \in \mathbb{N}\) and consider the two partitions

\[ \pi = < 0, 1/N, 2/N, \ldots, N - 1/N, 1 > \]

and

\[ \pi' = < 0, 1/2N, 3/2N, \ldots, 1 - 1/2N, 1 >. \]

We define \(I_k = \left[\frac{k-1}{N}, \frac{k}{N}\right]\) \((k = 1, \ldots, N)\) and \(I'_k = \left[\frac{k-1}{N} + \frac{1}{2N}, \frac{k}{N} + \frac{1}{2N}\right]\)
\((k = 1, \ldots, N - 1)\). The Lemma above suggest to consider for \(I = [a, b]\) the interval

\[ I(\delta) = [a + \delta, b - \delta]. \]
(for \(a = 0\) we will use \(I(\delta) = [0, b - \delta]\) and for \(b = 1\) \(I(\delta) = [a + \delta, 1]\) why?). Show that

\[
[0, 1] \subset \bigcup_{k=1}^{N} I_k(\delta) \cup \bigcup_{k=1}^{N-1} I'_k(\delta).
\]

(If you make a picture this is obvious).

(d) Let \(A_{\delta,k} = \{x \in [0, 1] : \omega_\delta(x) \geq \frac{1}{k}\}\). Let \(N \in \mathbb{N}\) such that \(\delta < 1/4N\).

Show that the set \(A_{\delta,k}\) is contained in intervals \(J_1, \ldots, J_m\) such that

\[
\sum_j |J_j| \leq k \left( \bar{S}(f, \pi) - S(f, \pi) \right) + k \left( \bar{S}(f, \pi') - S(f, \pi') \right).
\]

(e) Use the Cauchy-criterion for the Darboux integral to show that for a Darboux-integrable function

\[
\bigcap_\delta A_{\delta,k}
\]

has measure 0.

(f) Show that for a Darboux-integrable function the set of non-continuity points has measure 0.