Transition-hw3

Due date: September 27.

(1) Show the discrete Chebychev inequality: Let (a_n) be positive numbers such that

$$S = \sum_n a_n < \infty$$

is finite. Let $\lambda > 0$. Show that the set $A_\lambda = \{n \in \mathbb{N} : a_n \geq \lambda\}$ has not more than S/λ many elements.

(2) Let $A_1, ..., A_k$ be subset of $\{1, ..., n\}$ such that

$$\sum_k |A_k| < n$$

Show there is an element in $\bigcup_{j=1,...,k} A_k^c$. ($A_k^c = \{1, ..., n\} \setminus A_k$ is the complement.)

(3) Show that the countable union of sets of measure 0 has measure 0.

(4) We want to show that a bounded function is Riemann integrable if and only if the set of discontinuity points has measure 0. Let $f : [0, 1] \to \mathbb{R}$ be a function such that

$$\sup_{x \in [0,1]} |f(x)| \leq B .$$

We define

$$\omega_\delta(x) = \sup\{|f(y) - f(x)| : |x - y| < \delta, |x - z| < \delta\} .$$

(a) Show that $\omega_\delta(x) \leq 2B$.

(b) Let $I = [a, b]$ be an interval and $a + \delta \leq x \leq b - \delta$. Show that

$$\omega_\delta(x)(b - a) \leq (\sup_I f - \inf_I f)(b - a) .$$

(c) Let $N \in \mathbb{N}$ and consider the two partitions

$$\pi = < 0, 1/N, 2/N, ..., N - 1/N, 1 >$$

and

$$\pi' = < 0, 1/2N, 3/2N, ..., 1 - 1/2N, 1 > .$$

We define $I_k = [\frac{k-1}{N}, \frac{k}{N}]$ $(k = 1, ..., N)$ and $I'_k = [\frac{k-1}{N} + \frac{1}{2N}, \frac{k}{N} + \frac{1}{2N}]$ $(k = 1, ..., N - 1)$. The Lemma above suggest to consider for $I = [a, b]$ the interval

$$I(\delta) = [a + \delta, b - \delta]$$

(for $a = 0$ we will use $I(\delta) = [0, b - \delta]$ and for $b = 1$ $I(\delta) = [a + \delta, 1]$ why?). Show that

$$[0,1] \subset \bigcup_{k=1}^{N} I_k(\delta) \cup \bigcup_{k=1}^{N-1} I'_k(\delta) \ .$$

(If you make a picture this is obvious).

(d) Let $A_{\delta,k} = \{x \in [0,1] \ : \ \omega_\delta(x) \geq \frac{1}{k}\}$. Let $N \in \mathbb{N}$ such that $\delta < 1/4N$. Show that the set $A_{\delta,k}$ is contained in intervals $J_1, ..., J_m$ such that

$$\sum_j |J_j| \ \leq \ k \left(\bar{S}(f,\pi) - \underline{S}(f,\pi)\right) + k \left(\bar{S}(f,\pi') - \underline{S}(f,\pi')\right) \ .$$

(e) Use the Cauchy-criterion for the Darboux integral to show that for a Darboux- integrable function

$$\bigcap_\delta A_{\delta,k}$$

has measure 0.

(f) Show that for a Darboux- integrable function the set of non-continuity points has measure 0.