Transition-hw3

Due date: September 20.

(1) (a) Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. Show that f is Lipschitz if and only if
$$\sup_x |f'(x)|$$
is finite.

(b) Determine the maximal $b > 0$ such that $f(x) = x^2 - x$ has Lipschitz constant ≤ 1 on $[0, b]$.

(2) Let X be a metric space, $0 < c < 1$ and $f : X \to X$ such that
$$d(f(x), f(y)) \leq cd(x, y) \ .$$
Let $x_0 \in X$ and define inductively $x_{n+1} = f(x_n)$. Show that (x_n) is Cauchy. Study the function $f(x) = 1 - x$ on $[0, 1]$ and show that this does not work for $c = 1$.

(3) Let X be the completion of $(\mathbb{Z}, dd_p)$ and $y \in \mathbb{Z}$.

(a) Show that exists a continuous map $f : X \to X$ such that $f(x) = x + y$ for all $x \in \mathbb{Z}$.

(b) Show that there exists continuous map $add : X \times X \to X$ satisfying $add(x, y) = x + y$ for all $x, y \in \mathbb{Z}$. (Here the distance on $X \times X$ is given by
$$d((x_1, x_2), (y_1, y_2)) \ = \ d(x_1, y_1) + d(x_2, y_2) \ .)$$

(c) What can you say about multiplication? What structure do you expect for X.

(4) (a) Let X be a metric space and $f : X \to Y$ be continuous. Show that $f(K)$ is compact for all $K \subset X$ compact.

(b) Let X be compact metric space and $f : X \to Y$ be bijective continuous map. Show that f^{-1} is continuous.