Transition course -hw2

Due date: Monday, September 13

(1) Let $1 \leq p < \infty$. Show that $\ell_p = \{(x_n) \ : \ \left(\sum_n |x_n|^p\right)^{\frac{1}{p}}$ is a complete metric spaces with respect to

$$d_p(x, y) \ = \ \left(\sum_n |x_n - y_n|^p\right)^{\frac{1}{p}} \ .$$

(Hint: Use $d_p(x, y) \ = \ \lim_m \left(\sum_{n \leq m} |x_n - y_n|^p\right)^{\frac{1}{p}}$ in the proof of the triangle inequality.)

(2) Use the result from the lecture for a short proof of the fact

$$\ell_\infty = \{(x_n) \ : \ \sup_n |x_n| < \infty\}$$

is complete with respect to $d(x, y) = \sup_n |x_n - y_n|$.

(3) Show that $x_n = \sum_{j \leq n} p^j$ is a Cauchy sequence in $(\mathbb{Z}, dd_p)$. Show that there is no limit $x \in \mathbb{Z}$. (Hint use the unique decomposition $x = \sum a_k p^k$, $a_k \in \{0, ..., p-1\}$ for positive integers.)

(4) A function $f : X \to Y$ is uniformly continuous of

$$\forall_{\varepsilon > 0} \exists_{\delta > 0} \forall_{x, y \in X} (d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon) \ .$$

Show that $f : (0, 1) \to \mathbb{R}$, $f(x) = 1/x$ is not uniformly continuous.

(5) The space $C_b(X, \mathbb{R})$ is also a vector space over $\mathbb{R}$. Find n-linearly independent elements for
   (a) $X = \{1, ..., n\}$ with the discrete metric.
   (b) $X = [0, 1]$ with the usual metric. (Hint: polynomials?)