Transition-hw11

**Due Date:** Wednesday, December 8

(1) Let $F : C[0,1] \to C[0,1]$ be the function

$$F(x)(t) = \int\limits_0^t x(s)ds \,.$$

Calculate $F'(1)$ and show that $F'(1)$ has no bounded inverse. (Hint: That would imply $\sup_{0\le s\le 1} |h(s)| \le c \sup_t |\int_0^t h(s)ds|$ for com constant $c$. However, a clever choice such that $\int_0^t h(s)ds = \sin(nt)$ makes that pretty impossible).

(2) We want to apply the implicit function theorem for

$$F(x,y)(t) = \int_0^t x(s)y(s)ds$$

at $(x_0, y_0) = (1,1)$. Show that $D_2F(1,1)$ is not invertible and thus the implicit function theorem does not imply. Show that the solution to $F(x,y) = F(1,1)$ is given by $y(s) = \frac{1}{x(s)}$ and that this is well defined for $\|x - 1\| < 1$.

(3) A better way to obtain a good solution is to consider the function

$$F(x,y)(s) = x(s)y(s)$$

Show that the implicit function theorem applies and yields a map $u : B(1,\delta) \to C[0,1]$ such that

$$F(x, u(x)) = F(1,1) \,.$$

Calculate the derivative.

(4) Show that there exists a differentiable function $f : [1-\delta, 1+\delta] \to \mathbb{R}$ satisfying

$$(1 + f(x))^{\frac{1}{3}} = f(x)$$

Calculate the derivative at 1.