Practice problems for the final

(1) In the following problem you use $D = C[0,1]$ and $X = L_1[0,1]$ the completion of D with respect to the metric

$$d_1(f,g) \;=\; \int_0^1 |f-g|dt \;.$$

Show that the function $u : D \to \mathbb{R}$ given by

$$u(f) \;=\; \int_0^1 f(t)dt$$

has a unique extension on X. Let $g \in C[0,1]$ such that

$$\sup_s |g(s)| \le \alpha < 1$$

Is there a fixpoint f in D for $T(f)(s) = g(s)f(s)$? What happens if we use

$$g(s) \;=\; \begin{cases} \frac{1}{2} & 0 \le s \le \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} < s \le 1 \end{cases} \;.$$

(Then $T(D) \subset L_1$.)

Solution: We only have to show that u is uniformly continuous. For this it suffice to show that u is Lipschitz. Indeed,

$$|u(f) - u(h)| \;=\; |\int_0^1 [f(t) - h(t)]dt| \;\le\; \int_0^1 |f(t) - h(t)|dt \;=\; d_1(f,h) \;.$$

Thus the unique extension principle yields the claim. Not we consider $T : D \to D$ defined by $T(f) = fg$. Note that

$$d_1(T(f), T(h)) \;\le\; \int |g(t)||f(t) - h(t)|dt \;\le\; \sup |g(t)| d_1(f,h) \;\le\; \alpha d_1(f,h) \;.$$

Thus T is Lipschitz with constant $\le \alpha$. In particular, T admits an extension $T : L_1 \to L_1$ which is also Lipschitz with constant $\le \alpha$. By he contraction principle there exists a unique fixpoint f for T. This means

$$f(t)g(t) \;=\; f(t)$$

or equivalently $f(t)(1 - g(t)) \;=\; 0$. Thus $f(t) \;=\; 0$ is the only fixpoint-how boring. The argument in the second case is similar, we find $f \in L_1$ such that $T(f) = f$ but the fixpoint is unique and $f(t) = 0$ is a solution. So this is the only solution.

(2) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Calculate e^{xA} and solve the system of equations

$$\vec{y}'(x) = A(\vec{y}(x)) \quad , \quad \vec{y}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} .$$

Determine the JNF without finding a basis.

Solution: If we consider $(A-1)$ then it is east to see that $\dim\ker(A-1) = 1$, $\dim\ker(A-1)^2 = 2$, $\dim\ker(A-1)^3 = 3$ is $\dim\ker(A-1)^4 = 4$. Thus we have one Jordan block of size 4. We consider

$$B = (A-1) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then $B^2 = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and and $B^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. This yields

$$e^{xA} = e^x \left(\sum_{k=0}^{\infty} \frac{x^k B^k}{k!} \right) = e^t(id + xB + \frac{x^2}{2}B^2 + \frac{x^3}{3!}B^3 .$$

This means

$$e^{xA} = e^x \begin{pmatrix} 1 & x & x+\frac{x^2}{2} & x+x^2+\frac{x^3}{6} \\ 0 & 1 & x & x+\frac{x^2}{2} \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

For the system, we have $\vec{y}(t) = e^{xA}y(0)$-Plug it in.

(3) In this problem we want to solve the differential equation

pde (0.1)
$$\frac{\partial^2}{\partial x \partial t} F(x,t) = F(x,t) \quad , \quad F(0,t) = f(t)$$

for some continuous function f on $[0,1]$. Recall that for any Banach space X and a continuous linear map $T : X \to X$. The solution to

$$y'(x) \;=\; T(y(x)) \quad y(0) = y_0$$

is given by $y(x) \;=\; e^{xT}y_0$. (We have used that for matrices). The trick here is to introduce

$$T(f)(t) \;=\; \int\limits_0^t f(s)ds \;.$$

(1) Show that $g = T(f)$ if and only if $g(0) = 0$ and $g'(t) = f(t)$.
(2) Show that $T : C[0,1] \to C[0,1]$ is continuous.
(3) Let $f \in C[0,1]$. Let $n \in \mathbb{N}$. Differentiate the function

$$g_n(t) \;=\; \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)ds \;.$$

(4) Show that $T^n(f) = g$ if and only if $g^{(n)} = f$ and

$$g(0) = g'(0) = \cdots = g^{(n-1)}(0) = 0 \;.$$

Use this to show that

$$T^n(f) \;=\; \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)ds \;.$$

(5) Show that

$$F(x,t) \;=\; f(t) + \sum_{n=1}^{\infty} \frac{x^n}{n!} \int\limits_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)ds$$

is a solution of (pde0.1). (Hint F is a such a solution iff $\frac{d}{dx}F(x,t) \;=\; \int_0^t F(x,s)ds$ and we have just found a solution to that, isn't it.)

Solution: a) is just the fundamental theorem of calculus. b) We have

$$\|T(f)\| \;=\; \sup_t \left| \int_0^t f(s)ds \right| \;\le\; \sup_t t \sup_s |f(s)| \;\le\; \sup_s |f(s)| \;.$$

This means $\|T\| \le 1$. (T is Lipschitz with constant 1.) c) is an application of the chain rule (we assume $n > 1$)

$$G(t,r) \;=\; \int_0^t \frac{(r-s)^{n-1}}{(n-1)!} f(s)ds$$

$\frac{\partial G}{\partial t} \;=\; 0$ and $\frac{\partial G}{\partial r} = \int_0^t \frac{(r-s)^{n-2}}{(n-2)!} f(s)ds$. Now, we use

$$g_n(t) \;=\; G(t,t)$$

and deduce the assertion. d) That is induction and we note from c) that g_n satisfies the requirements, thus $T^n(f) = g_n$. e) We known that

$$y(x) \;=\; e^{xT}(y_0)$$

is the unique solution to

$$y'(x) \;=\; T(y(x)) \quad , \quad y(0) = y_0 \,.$$

We put $y_0 = f \in C[0,1]$ and deduce that

$$y(x) \;=\; e^{xT}(f)$$

is a solution. We define $F(x,t) \;=\; e^{xT}(f)(t)$. Since point evaluation is continuous we deduce

$$\frac{\partial}{\partial x} F(x,t) \;=\; T(F(x,t)) \;=\; \int_0^t F(x,s)ds \,.$$

We differentiate another time with respect to t and get

$$\frac{\partial^2}{\partial t \partial x} F(x,t) \;=\; F(x,t)ds \,.$$

There you go.

(4) Consider $Y = \ell_2$ and $F : \ell_2 \times \ell_2 \to \ell_2$ given by

$$F((x_n),(y_n)) \;=\; y_n - x_n^2 y_n - \frac{2}{n^4} \,.$$

We consider the couple $(x^0, y^0) = ((\frac{2}{n^2}),(0))$.

(1) Calculate D_1F and D_2F. It is easy to see that these maps are continuous (try).

(2) Show that every linear map $L : \ell_2 \to \ell_2$ given by $L(h_n) \;=\; (a_n h_n)$ such that $C = \sup_n |a_n|^{-1}$ is finite satisfies

$$\|L^{-1}\| \;\leq\; C \,.$$

(3) Show that $D_2F(x^0, y^0)$ is invertible.

(4) Show that there exists a $\delta > 0$ and function $f : B(x^0, \delta) \to \ell_2$ such that

$$F(x, f(x)) = 0$$

for all $x \in B(x^0, \delta)$. Calculate

$$f'(x^0) \in L(\ell_2) \,.$$

Solution: We consider

$$\begin{aligned}
F((x_n+h_n),(y_n+k_n)) - F(x_n,y_n)_n \;&=\; k_n-(x_n+h_n)^2(y_n+k_n)+x_n^2y_n \\
&= k_n-[(x_n^2+h_n^2+2x_nh_n)(y_n+k_n)]+x_n^2y_n \\
&= k_n-[(x_n^2y_n+h_n^2y_n+2x_ny_nh_n)+(x_n^2k_n+h_n^2k_n+2x_nh_nk_n)]+x_n^2y_n \\
&= k_n-2x_ny_nh_n-x_n^2k_n-(h_n^2y_n+h_n^2k_n+2x_nh_nk_n)\ .
\end{aligned}$$

Note that

$$\|(h_n^2y_n)\|_2 \;\le\; \sup_n |y_n|(\sum_n |h_n|^4)^{\frac{1}{2}} \;\le\; \sup_n |y_n|(\sum_n |h_n|^2)^2\ .$$

Similarly,

$$\begin{aligned}
\|(2x_nh_nk_n)\|_2 &\le \sup_n |x_n|(\sum_n |h_nk_n|^2)^{\frac{1}{2}} \;\le\; \sup_n |x_n|[(\sum_n |h_n|^4)^{1/2}(\sum_n |k_n|^4)^{1/2}]^{1/2} \\
&\le \sup_n |x_n|\|(h_n)\|_2\|(k_n)\|_2 \\
&\le \sup_n |x_n|\frac{\|(h_n)\|_2^2+\|(k_n)\|_2^2}{2}\ .
\end{aligned}$$

This is fine, too. The last term is similar. Thus F is differentiable and

$$D_1F((x_n)(y_n))(h_n) \;=\; (-2x_ny_nh_n)_n$$

and

$$D_2F((x_n)(y_n))(k_n) \;=\; ((1-x_n^2)k_n)_n\ .$$

(2) is obvious the inverse map is given by $L(h_n)=(a_n^{-1}h_n)$ and

$$\|(a_n^{-1}h_n)\|_2 \le \sup_n |a_n^{-1}|\|(h_n)\|_2\ .$$

For (3) we note that $a_n = 1-\frac{2}{n^2}$ is bounded away from 0. Finally (4) is easy

$$f'(x^0)(h_n) \;=\; (a_n^{-1}(-\frac{2}{n^2}0h_n)) \;=\; 0\ .$$

Thus $f(x^0)=0$.