1. Continuous functions between metric spaces

Continuous functions 'preserve' properties of metric spaces and allow to describe deformation of one metric space into another. There are three different (but equivalent) ways of defining continuity, the ε-δ-criterion, the sequence criterion and the topological criterion. Each of them is interesting in its own right.

Definition 1.1. *Let (X, d) and (Y, d') be metric spaces. A map $f : X \to Y$ is called continuous if for every $x \in X$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that*

edelt (1.1)
$$d(x, y) < \delta \quad \Longrightarrow \quad d'(f(x), f(y)) < \varepsilon \, .$$

Let us use the notation

$$B(x, \delta) \;=\; \{y \,:\, d(x, y) < \delta\} \, .$$

For a subset $A \subset X$, we also use the notation

$$f(A) \;=\; \{f(x) \,:\, x \in A\} \, .$$

Similarly, for $B \subset Y$

$$f^{-1}(B) \;=\; \{x \in X \,:\, f(x) \in B\} \, .$$

Then (edelt 1.1) means

$$f(B(x, \delta)) \;\subset B(f(x), \varepsilon) \, .$$

Or in a very non-formal way

f maps small balls into small balls .

Our aim is to prove a criterion for continuity in terms of so called open sets. This criterion illustrates simultaneously the role of open sets and its interaction with continuity and has a genuinely geometric flavor.

Definition 1.2. *A subset O of a metric space is called open if*

$$\forall x \in O \;: \exists \delta > 0 \;: B(x, \delta) \subset O \, .$$

Examples:

$$O \;=\; (-1, 1) \, , \; O \;=\; \mathbb{R} \, , \; O \;=\; (-1, 1) \times (-2, 2)$$

are open in $\mathbb{R}$, $(\mathbb{R}^2, d_2)$ respectively.

Remark 1.3. *The sets $B(x, \varepsilon)$, $x \in X$, $\varepsilon > 0$ are open.*

Proposition 1.4. *Let (X, d), (Y, d') be metric spaces and $f : X \to Y$ be a map. f is contiuous iff $f^{-1}(O)$ is open for all open subsets $O \subset Y$.*

PROOF. $\Rightarrow$: We assume that f is continuous and O is open. Let $x \in f^{-1}(O)$, i.e. $f(x) \in O$. Since O is open, there exists an $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subset O$. By continuity, there exists a $\delta > 0$ such that

$$f(B(x, \delta)) \subset B(f(x), \varepsilon) \subset O \,.$$

Therefore

$$B(x, \delta) \subset f^{-1}(O) \,.$$

Since $x \in f^{-1}(O)$ was arbitrary, we deduce that $f^{-1}(O)$ is open.
$\Leftarrow$: Let $x \in X$ and $\varepsilon > 0$. Let us show that

$$B(f(x), \varepsilon)$$

is a on open subset of (Y, d'). Indeed, let $y \in B(f(x), \varepsilon)$ define $\varepsilon' = \varepsilon - d'(y, f(x))$. Let $z \in Y$ such that

$$d(z, y) < \varepsilon'$$

then

$$d(f(x), z) \;\leq\; d(f(x), y) + d(y, z) \;<\; d(f(x), y) + \varepsilon - d'(y, f(x)) \;=\; \varepsilon \,.$$

Thus

$$B(y, \varepsilon - d'(f(x), y)) \;\subset B(f(x), \varepsilon) \,.$$

By the assumption, we see that $f^{-1}(B(f(x), \varepsilon))$ is an open set. Since $x \in f^{-1}(B(f(x), \varepsilon))$, we can find a $\delta > 0$ such that

$$B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon)) \,.$$

Hence, for all $\tilde{x}$ with $d(x, \tilde{x}) < \delta$, we have

$$d'(f(x), f(\tilde{x})) < \varepsilon \,.$$

The assertion is proved. ∎

Examples:

(1) Let (X, d) be a metric space and $x_0 \in X$ be a point , then $f(x) = d(x, x_0)$ is continuous. Indeed, the triangle inequality implies

$$d(d(x, x_0), d(d(y, 0)) \;=\; |d(x, x_0) - d(y, x_0)| \;\leq\; d(x, y)$$

This easily implies the assertion.

(2) On $\mathbb{R}^n$ with the standard euclidean metric $d = d_2$, the function $f : \mathbb{R}^n \to \mathbb{R}^n$ defined by $f(x) = d(x, 0)x$ is continuous.

(3) (Exercise) The function $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (\cos(x_1), \sin(x_2), \cos(x_1))$ is continuous.

DEFINITION 1.5. *Let (X, d), (Y, d') be a metric space. The space $C(X, Y)$ is the set of all continuous functions from X to Y. Let $x_0 \in X$ be a point. Then*

$$C_b(X,Y) = \{f : X \to Y : f \text{ is continuous and } \sup_{x\in X} d'(f(x), f(x_0)) < \infty\}$$

is the subset of bounded continuous functions.

PROPOSITION 1.6. *Let (X, d), (Y, d') be metric spaces and $x_0 \in X$. Then $C_b(X, Y)$ equipped with*

$$d(f, g) = \sup_{x\in X} d'(f(x), g(x))$$

is a metric space.

Problem: Show that d is not well-defined on $C(\mathbb{R}, \mathbb{R})$.

Proof: $d(f, g) = 0$ if and only if $f(x) = g(x)$ for all $x \in X$. This means $f = g$. Let us show that d is well-defined. Indeed, if $f, g \in C_b(X, Y)$. Then

$$\begin{aligned}\sup_x d'(f(x), g(x)) &\le \sup_x d'(f(x), f(x_0)) + d'(f(x_0), g(x_0)) + d'(g(x_0), g(x)) \\ &\le \sup_x d'(f(x), f(x_0)) + d'(f(x_0), g(x_0)) + \sup_x d(g(x_0), g(x))\end{aligned}$$

is finite. Let h be a third function and $x \in X$. Than

$$d'(f(x), g(x)) \le d'(f(x), h(x)) + d(h(x), g(x)) \le d(f, h) + d(h, g) .$$

Taking the supremum yields the assertion. ■

alg PROPOSITION 1.7. *Let (X, d) be a metric space. Then $C(X, \mathbb{R})$ is closed under (pointwise-) sums, products and multiplication with real numbers. ($C(X, \mathbb{R})$ is an algebra over $\mathbb{R}$).*

REMARK 1.8. *Let $X = \mathbb{N}$ and $d(x, y) = 1$ of $x \neq y$ and $d(x, y) = 0$ for $x = y$. (This is called the discrete metric). Then $C(X, \mathbb{R})$ is an infinite dimensional vector space.*

PROOF OF alg 1.7. Let $f, g \in C(X, \mathbb{R})$ be continuous and $x \in X$. Consider $x' \in X$. Then

$$\begin{aligned}fg(x) - fg(y) &= f(x)g(x) - f(y)g(y) = (f(x) - f(y))g(x) + f(y)(g(x) - g(y)) \\ &= (f(x) - f(y))g(x) + f(x)(g(x) - g(y)) + (f(y) - f(x))(g(x) - g(y)) .\end{aligned}$$

Let $\varepsilon > 0$ and $\tilde{\varepsilon} = \min\{\varepsilon, 1\}$. We may choose $\delta_1 > 0$ such that

$$d(f(x), f(y))(1 + |g(x)|) < \frac{\tilde{\varepsilon}}{3}$$

holds for all $d(x,y) < \delta_1$. Similarly, we may choose $\delta_2 > 0$ such that

$$d(g(x), g(y))(1 + |f(x)|) < \frac{\tilde{\varepsilon}}{3} .$$

Let $\delta = \min(\delta_1, \delta_2)$ and $d(x,y) < \delta$. Then we deduce that

$$d(fg(x), fg(y)) = |fg(x) - fg(y)| < \frac{\tilde{\varepsilon}}{3} + \frac{\tilde{\varepsilon}}{3} + \frac{\tilde{\varepsilon}^2}{9} < \tilde{\varepsilon} \le \varepsilon .$$

Thus fg is again continuous. The other assertions are easier. ■

COROLLARY 1.9. *The polynomials on $\mathbb{R}$ are continuous.*

LEMMA 1.10. *Let $1 \le p \le \infty$ and $x, y \in \mathbb{R}^n$, then*

$$\frac{1}{n^{\frac{1}{p}}} d_p(x,y) \le d_\infty(x,y) \le d_p(x,y) .$$

PROOF. The last inequality is obvious. For the first one, we consider $x, y \in \mathbb{R}^n$ and $1 \le p < \infty$, then by estimating every element in the sum against the maximum

$$d_p(x,y)^p = \sum_{i=1}^n |x_i - y_i|^p \le n \max\{|x_i - y_i|^p\} .$$

Taking the p-th root, we deduce the assertion. ■

COROLLARY 1.11. *Let $1 \le p, q \le \infty$, then the identity map $id : (\mathbb{R}^n, d_p) \to (\mathbb{R}^n, d_q)$ is continuous.*

PROOF. We have for all $x \in \mathbb{R}^n$ and $\varepsilon > 0$

$$B_{d_p}(x, \frac{\varepsilon}{n}) \subset B_{d_q}(x, \varepsilon) .$$

This easily implies the assertion. ■

COROLLARY 1.12. *The metrics d_p define the same open sets on $\mathbb{R}^n$.*

DEFINITION 1.13. *Let (X, d) be a metric space. We say that a sequence (x_n) converges to x if for all $\varepsilon > 0$ there exists n_0 such that for $n > n_0$ we have*

$$d(x_n, x_0) < \varepsilon .$$

In this case we write

$$\lim_n x_n = x$$

or more explicitly

$$d - \lim_n x_n = x .$$

A sequence (x_n) is convergent, if there exists $x \in X$ with $\lim_n x_n = x$.

Examples: $d_2 - \lim_n \frac{1}{n} = 0$, $dd_3 - \lim_n 3^n = 0$. (What axioms of the natural numbers are involved?).

PROPOSITION 1.14. *Let (X, d), (Y, d') be metric spaces and $f : X \to Y$ be a map. Then f is continuous if for every convergent sequence (x_n) in X*

$$\lim_n f(x_n) \;=\; f(\lim_n x_n)\;.$$

Proof: $\Rightarrow$: Let $x = \lim_n x_n$ and $\varepsilon > 0$, then there exists a $\delta > 0$ such that

$$d(y, x) < \delta \Rightarrow d'(f(y), f(x)) < \varepsilon\;.$$

Let $n_0 \in \mathbb{N}$ be such that

$$d(x_n, x) < \delta$$

for all $n > n_0$, then

$$d'(f(x_n), f(x)) < \varepsilon$$

for all $n > n_0$. Hence

$$\lim_n f(x_n) \;=\; f(x)\;.$$

$\Leftarrow$ Let $x \in X$ and assume in the contrary that

$$\exists \varepsilon > 0\, \forall \delta > 0 \exists y\ : d(y, x) < \delta \text{ and } d'(f(x), f(y)) \geq \varepsilon\;.$$

Applying these successively for all $\delta = \frac{1}{k}$, we find a sequence (x_k) such that

$$d(x_k, x) \;<\; \frac{1}{k} \quad \text{and} \quad d'(f(x_k), f(x)) \geq \varepsilon'\;.$$

and thus

$$\lim_k x_k \;=\; x\;.$$

By assumption, we have

$$\lim_k f(x_k) \;=\; f(x)\;.$$

Hence, there exists a k_0 such that for all $k > k_0$

$$d(f(x_k), f(x)) < \varepsilon\;.$$

a contradiction. ■