CHAPTER 1

Metric Spaces

1. Definition and examples

Metric spaces generalize and clarify the notion of distance in the real line. The definitions will provide us with a useful tool for more general applications of the notion of distance:

DEFINITION 1.1. *A metric space is given by a set X and a distance function $d : X \times X \to \mathbb{R}$ such that*

i) *(Positivity) For all $x, y \in X$*

$$0 \leq d(x,y) \ .$$

ii) *(Non-degenerated) For all $x, y \in X$*

$$0 \ = \ d(x,y) \quad \Leftrightarrow \quad x \ = \ y \ .$$

iii) *(Symmetry) For all $x, y \in X$*

$$d(x,y) \ = \ d(y,x)$$

iv) *(Triangle inequality) For all $x, y, z \in X$*

$$d(x,y) \ \leq \ d(x,z) + d(z,y) \ .$$

Examples:

i) $X = \mathbb{R}$, $d(x,y) = |x - y|$.

ii) $X = \mathbb{R}^2 \ = \ \mathbb{R} \times \mathbb{R}$, $x = (x_1, x_2)$, $y = (y_1, y_2)$

$$d_1(x,y) \ = \ |x_1 - y_1| + |x_2 - y_2| \ .$$

iii) $X = \mathbb{R}^2$, $x = (x_1, x_2)$, $y = (y_1, y_2)$

$$d_2(x,y) \ = \ \left(|x_1 - y_1|^2 + |x_2 - y_2|^2\right)^{\frac{1}{2}} \ .$$

iv) Let $X = \{p_1, p_2, p_3\}$ and

$$d(p_1,p_2) \ = \ d(p_2,p_1) \ = \ 1 \ ,$$

$$d(p_1,p_3) \ = \ d(p_3,p_1) \ = \ 2 \ ,$$

$$d(p_2, p_3) \;=\; d(p_3, p_2) \;=\; 3\,.$$

Can you find a triangle (p_1, p_2, p_3) in the plane with these distances?

v) Let $X = \{p_1, p_2, p_3\}$ and

$$\begin{aligned} d(p_1, p_2) &= d(p_2, p_1) \;=\; 1\,, \\ d(p_1, p_3) &= d(p_3, p_1) \;=\; 2\,, \\ d(p_2, p_3) &= d(p_3, p_2) \;=\; 4\,. \end{aligned}$$

Can you find a triangle (p_1, p_2, p_3) in the plane with these distances?

vi) The French railway metric (Chicago suburb metric) on $X = \mathbb{R}^2$ is defined as follows: Let $x_0 = (0, 0)$ be the origin, then

$$d_{SNCF}(x, y) \;=\; \begin{cases} d_2(x, y) & \text{if there exists a } t \in \mathbb{R} \text{ such that } x_1 = ty_1 \\ & \text{and } x_2 = ty_2 \\ & \\ d_2(x, x_0) + d_2(x_0, y) & \text{else} \end{cases} \,.$$

Exercise: Show that the railroad metric satisfies the triangle inequality.

It is by no means trivial to show that d_2 satisfies the triangle inequality. In the following we write $0 = (0, ..., 0)$ for the origin in $\mathbb{R}^n$.

CS LEMMA 1.2. *Let $x, y \in \mathbb{R}^n$, then*

$$|\sum_{i=1}^n x_i y_i| \;\le\; \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^n |y_i|^2\right)^{\frac{1}{2}}$$

LEMMA 1.3. *On $\mathbb{R}^n$ the metric*

$$d_2(x, y) \;=\; \left(\sum_{i=1}^n |x_i - y_i|^2\right)^{\frac{1}{2}}$$

satisfies the triangle inequality.

PROOF. Let $x, y, z \in \mathbb{R}^n$. Then we deduce from Lemma CS 1.2

$$\begin{aligned} d(x, y)^2 &= \sum_{i=1}^n |x_i - y_i|^2 \;=\; \sum_{i=1}^n |(x_i - z_i) - (y_i - z_i)|^2 \\ &= \sum_{i=1}^n |(x_i - z_i)|^2 - 2\sum_{i=1}^n (x_i - z_i)(y_i - z_i) + \sum_{i=1}^n |y_i - z_i|^2 \\ &\le d(x, z)^2 + 2d(x, y)d(y, z) + d(y, z) \end{aligned}$$

$$= \quad (d(x,z)+d(y,z))^2 \, .$$

Hence,

$$d(x,y) \;\le\; d(x,z)+d(y,z)$$

and the assertion is proved. ■

More examples:

(1) Let n be a prime number. On $\mathbb{Z}$ we define

$$dd_n(x,y) \;=\; n^{-\max\{m\in\mathbb{N}\,:\, m \text{ divides x-y}\}} \, .$$

The n-adic metric satisfies a stronger triangle inequality

$$dd_n(x,y) \;\le\; \max\{dd_n(x,z), dd_n(z,y)\} \, .$$

(2) Let $1 \le p < \infty$. Then

$$d_p(x,y) \;=\; \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}$$

defines a metric n $\mathbb{R}^n$.

(3) For $p = \infty$

$$d_\infty(x,y) \;=\; \max_{i=1,..,n} |x_i - y_i|$$

also defines a metric on $\mathbb{R}^n$.

Project 1: Let $1 < p, q < \infty$ such that $1/p + 1/q = 1$. Show Minkowski's inequality.

Mink (1.1)
$$xy \le \frac{x^p}{p} + \frac{y^q}{q}$$

holds for all $x, y > 0$. **Hint:** the function $f(x) = -\ln x$ is convex on $(0,\infty)$.

Proof of the triangle inequality for d_p. The triangle inequality for $p = 1$ is obvious. We will fist show

mink2 (1.2)
$$|\sum_{i=1}^n x_i y_i| \;\le\; \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}$$

whenever $\frac{1}{p} + \frac{1}{q} = 1$. Let $t > 0$. We first observe that

$$|\sum_{i=1}^n x_i y_i| = \sum_{i=1}^n |tx_i||t^{-1}y_i| \;\le\; \sum_{i=1}^n \frac{1}{p}|tx_i|^p + \frac{1}{q}|t^{-1}y_i|^q$$
$$= \frac{t^p}{p} \sum_{i=1}^n |x_i|^p + \frac{t^{-q}}{q} \sum_{i=1}^n |y_i|^q \, .$$

What is best choice of t? Make

$$t^p \sum_{i=1}^{n} |x_i|^p \ = \ t^{-q} \sum_{i=1}^{n} |y_i|^q$$

i.e.

$$t^{p+q} \ = \ \frac{\sum_{i=1}^{n} |y_i|^q}{\sum_{i=1}^{n} |x_i|^p} \ .$$

This yields

$$
\begin{aligned}
|\sum_{i=1}^{n} x_i y_i| \leq t^p \sum_{i=1}^{n} |x_i|^p \ &= \ \frac{\left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{p}{p+q}}}{\left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{p}{p+q}}} \sum_{i=1}^{n} |x_i|^p \\
&= \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} |x_i|^p\right)^{1-\frac{1}{q}}
\end{aligned}
$$

Now, we proof the triangle inequality. Let $x = (x_i)$, (y_i) and $z = (z_i)$ in $\mathbb{R}^d$. Then we apply (mink2 1.2)

$$
\begin{aligned}
d_p(x,y)^p = \sum_{i=1}^{d} |x_i - y_i|^p \ &\leq \ \sum_{i=1}^{d} |x_i - y_i|^{p-1} (|x_i - z_i| + |z_i - y_i|) \\
&\leq \sum_{i=1}^{d} |x_i - y_i|^{p-1} |x_i - z_i| + \sum_{i=1}^{d} |x_i - y_i|^{p-1} |z_i - y_i| \\
&\leq \left(\sum_{i=1}^{d} (|x_i - y_i|^{p-1})^q\right)^{\frac{1}{q}} \left(\left(\sum_{i=1}^{d} |z_i - x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{d} |z_i - y_i|^p\right)^{\frac{1}{p}} \right) \ .
\end{aligned}
$$

However, $1 = 1/p + 1/q$ implies $p - 1 = p/q$ and thus $q(p-1) = p$. Hence we get

$$d_p(x,y)^p \ \leq \ d_p(x,y)^{p-1} (d_p(x,z) + d_p(z,y)) \ .$$

If $x \neq y$ we may divide and deduce the assertion. ■

2. Closed and Compact Sets

Let (X, d) be a metric space. We will say that a subset $A \subset X$ is *closed* if $X \setminus A$ is open.

closed PROPOSITION 2.1. *Let (X, d) be a complete metric space and $C \subset X$ a subset. C is closed iff every Cauchy sequence in C converges to an element in C.*

Proof: Let us assume C is closed and that (x_n) is a Cauchy sequence with elements in C. Let $x = \lim_n x_n$ be te limit and assume $x \notin C$. Since $X \setminus C$ is open

$$B(x, \varepsilon) \subset X \setminus C$$

for some $\varepsilon > 0$. Then there exists an n_0 such that $d(x_n, x) < \varepsilon$ for $n > n_0$. In particular,

$$x_{n_0+1} \in B(x, \varepsilon)$$

and thus $x_{n_0+1} \notin C$, a contradiction.
Now, we assume that every Cauchy sequence with values in C converges to an element in C. If $X \setminus C$ is not open, then there exists an $x \notin C$ and no $\varepsilon > 0$ such that

$$B(x, \varepsilon) \subset X \setminus C \, .$$

I.e. for every $n \in \mathbb{N}$, we can find $x_n \in C$ such that

$$d(x, x_n) < \frac{1}{n} \, .$$

Hence, $\lim x_n = x \in C$ but $x \notin C$, contradiction. ■
The most important notion in this class is the notion of compact sets. We will say that a subset $C \subset X$ is *compact* if For every collection (O_i) of open sets such that

$$C \subset \bigcup_i O_i \; = \; \{x \in X \mid \exists_{i \in I} x \in O_i\}$$

There exists $n \in \mathbb{N}$ and $i_1, ..., i_n$ such that

$$C \subset O_{i_1} \cup \cdots \cup O_{i_n} \, .$$

In other words

Every open cover of C has a finite subcover .

DEFINITION 2.2. *Let $X \subset \bigcup O_i$ be an open cover. Then we say that (V_j) is an open subcover if*

$$X \subset \bigcup_j V_j$$

all the V_j are open and for every j there exists an i such that

$$V_j \subset O_i \ .$$

It is impossible to explain the importance of 'compactness' right away. But we can say that there would be no discipline 'Analysis' without compactness. The most clarifying idea is contained in the following example.

PROPOSITION 2.3. *The set $[0,1] \subset \mathbb{R}$ is compact.*

Proof: Let $[0,1] \subset \bigcup_i O_i$. For every $x \in [0,1]$ there exists an $i \in I$ such that

$$x \in O_i \ .$$

Since O_i is open, we can find $\varepsilon > 0$ such that

$$x \in B(x,\varepsilon) \subset O_i \ .$$

Using the axiom of choice, we fine a function ε_x and i_x such that

$$x \in B(x,\varepsilon_x) \subset O_{i_x} \ .$$

Let us define the relation $x \preceq y$ if $x < y$ and

$$y - x \ \leq \ e_x + e_y \ .$$

The crucial point here is to define

$$S \ = \ \{x \in [0,1] \mid \exists\, x_1,...,x_n \ : \frac{1}{2} \preceq x_1 \preceq \cdots \preceq x_n \preceq x\} \ .$$

We claim a) $\sup S \in S$ and b) $\sup S = 1$.
Ad a): Let $y = \sup S \in [0,1]$. Then there exists an $x \in S$ with

$$y - \varepsilon_y < x \leq y \ .$$

Then obviously $x \preceq y$. Since $x \in S$, we can find

$$\frac{1}{2} \preceq x_1 \preceq \cdots \preceq x_n \preceq x \preceq y \ .$$

Thus $y \in S$.
Ad b): Assume $y = \sup S < 1$. Let $0 < \delta \ = \ \min(e_y, 1-y)$. Then

$$y + \delta - y \ = \ \delta \ \leq \ \varepsilon_y + \varepsilon_{y+\delta} \ .$$

By a), we find

$$\frac{1}{2} \preceq x_1 \preceq \cdots \preceq x_n \preceq y \preceq y + \delta$$

and thus $y + \delta \in S$. Contradiction to the definition of the supremum.
Assertion a) and b) are proved.

Therefore we conclude $1 \in S$ and thus find $x_1, ..., x_n$ such that

$$\frac{1}{2} \preceq x_1 \preceq \cdots \preceq x_n \preceq 1 \,.$$

Let $x_0 = \frac{1}{2}$ and $x_{n+1} = 1$, then by definition of $\preceq$, we have

$$[x_j, x_{j+1}] \subset B(x_j, \varepsilon_{x_j}) \cup B(x_{j+1}, \varepsilon_{x_{j+1}}) \subset O_{i_{x_j}} \cup O_{i_{x_{j+1}}}$$

for $j = 0, .., n$. Thus, we deduce

$$[\frac{1}{2}, 1] \subset \bigcup_{j=0}^{n} [x_j, x_{j+1}] \subset \bigcup_{j=0}^{n+1} O_{i_{x_j}} \,.$$

Doing the same trick with $[0, \frac{1}{2}]$, we find

$$[0, 1] \subset \bigcup_{j=0}^{m+1} O_{i_{x'_j}} \cup \bigcup_{j=0}^{n+1} O_{i_{x_j}}$$

and we have found our finite subcover. ■

subcom PROPOSITION 2.4. *Let $B \subset X$ be closed set and $C \subset X$ be a compact set, then*

$$B \cap C$$

is compact

Proof: Let $B \cap C \subset \bigcup O_i$ be an open cover. then

$$C \subset (X \setminus B) \cup \bigcup_i O_i$$

is an open cover for C, hence we can find a finite subcover

$$C \subset (X \setminus B) \cup O_{i_1} \cup \cdots \cup O_{i_n} \,.$$

Thus

$$B \cap C \ \subset O_{i_1} \cup \cdots \cup O_{i_n}$$

is a finite subcover. ■

ccc **LEMMA 2.5.** *Let (X,d) be a metric space and $D \subset X$ be a countable dense set in X, then for every subset $C \subset X$ and every open cover*

$$C \subset \bigcup_i O_i$$

we can find a countable subcover of balls.

Proof: Let us enumerate D as $D = \{d_n \,|\, n \in \mathbb{N}\}$. Let $x \in C$ and find $i \in I$ and $\varepsilon > 0$ such that

$$x \in B(x,\varepsilon) \subset O_i \, .$$

Let $k > \frac{2}{\varepsilon}$. By density, we can find an $n \in \mathbb{N}$ such that

$$d(x,d_n) < \frac{1}{2k} \, .$$

Then

$$x \in B(d_n, \frac{1}{2k}) \subset B(x, \frac{1}{k}) \subset B(x,\varepsilon) \subset O_i \, .$$

Let us define

$$M \;=\; \{(n,k) \mid \exists_{i\in I} B(d_n, \frac{1}{2k}) \subset O_i\} \, .$$

Then $M \subset \mathbb{N}^2$ is countable and hence there exists a map $\phi : \mathbb{N} \to M$ which is surjective (=onto). Hence for $V_m = B(d_{\phi_1(m)}, \frac{1}{2\phi_2(m)})$, ϕ_1, ϕ_2 the 2 components of ϕ we have

$$C \subset \bigcup_m V_m$$

and (V_m) is a countable subcover of balls of the original cover (O_i). ■

main **THEOREM 2.6.** *Let (X,d) be a metric space and D subset X be countable dense subset. Let $C \subset X$ be a subset. Then the following are equivalent*

i) a) *Every Cauchy sequence of elements in C converges to a limit in C.*
   b) *For every $\varepsilon > 0$ there exists points $x_1, ..., x_n \in X$ such that*

$$C \subset B(x_1,\varepsilon) \cup \cdots \cup B(x,\varepsilon) \, .$$

ii) *Every sequence in C has a convergent subsequence.*

iii) *C is compact.*

Proof: *i)* $\Rightarrow$ *ii)*. Let (x_n) be a sequence. Inductively, we will construct infinite subset $A_1 \supset A_2 \supset A_3 \cdots$ and y_1, y_2, y_3,... in X such that

$$\forall_{l \in A_j} \; : \; d(x_l, y_j) < 2^{-j-1} \, .$$

Put $A_0 = \mathbb{N}$. Let us assume $A_1 \supset A_2 \supset \cdots A_n$ and $y_1, ..., y_n$ have been constructed. We put $\varepsilon = 2^{-n-2}$ and apply condition $i)b)$ to find $z_1, ..., z_m$ such that

$$C \ \subset \ B(z_1,\varepsilon) \cup \cdots \cup B(z_m,\varepsilon) \ .$$

We claim that there must be a $1 \leq k \leq m$ such that

$$A_n(k) \ = \ \{l \in A_n \,|\, x_n \in B(z_k,\varepsilon)\}$$

has infinitely many elements. Indeed, we have

$$A_n(1) \cup \cdots \cup A_n(m) \ = \ A_n \ .$$

If they were all finite, then a finite union of finite sets would have finitely many elements. However A_n is infinite. Contradiction! Thus, we can find a k with $A_n(k)$ infinite and put $A_{n+1} = A_n(k)$ and $y_{n+1} = z_k$. So the inductive procedure is finished. Now, we can find an increasing sequence (n_j) such that $n_j \in A_j$ and deduce

$$d(x_{n_j}, x_{n_{j+1}}) \ \leq \ d(x_{n_j}, y_j) + d(y_j, x_{n_{j+1}}) < \frac{1}{2}2^{-j} + \frac{1}{2}2^{-j} \ = \ 2^{-j}$$

because $n_j \in A_j$ and $n_{j+1} \in A_{j+1} \subset A_j$. Thus (x_{n_j}) is Cauchy. Indeed, be induction, we deduce for $j < m$ that

$$\begin{aligned} d(x_{n_j}, x_{n_m}) \ &\leq \ d(x_{n_j}, x_{n_{j+1}}) + d(x_{n_{j+1}}, x_{n_{j+2}}) \cdots d(x_{n_{m-1}}, x_{n_m}) \\ &\leq \ 2^{-j} \sum_{k=0}^{m-1} 2^{-k} \ = \ 2^{1-j} \ . \end{aligned}$$

This easily implies the Cauchy sequence condition. By a) it converges to some $x \in C$. We got our convergent subsequence.

$ii) \Rightarrow iii)$: By Lemma 2.5, we can assume that

$$C \subset \bigcup_k O_k$$

and O_k's open. If we can find an n such that

$$C \subset O_1 \cup \cdots \cup O_n$$

the assertion is proved. Assume that is not the case and choose for every $n \in \mathbb{N}$ an $x_n \in C \setminus O_1 \cup \cdots \cup O_n$. According to the assumption, we have a convergent subsequence, i.e. $\lim_k x_{n_k} = x \in C$. Then $x \in O_{n_0}$ for some n_0 and there exists a $\varepsilon > 0$ such that

$$B(x,\varepsilon) \ \subset O_{n_0} \ .$$

By convergence, we find a k_0 such that $d(x, x_{n_k}) < \varepsilon$ for all $k > k_0$. In particular, we find a $k > k_0$ such that $n_k > n_0$. Thus

$$x_{n_k} \in B(x, \varepsilon) \in O_{n_0} \subset O_1 \cup \cdots \cup O_{n_k} .$$

Contradicting the choice of the (x_n)'s. We are done.
iii) $\Rightarrow$ *i)b)* Let $\varepsilon > 0$ and then

$$C \subset \bigcup_{x \in C} B(x, \varepsilon) .$$

thus a finite subcover yields *b)*.
iii) $\Rightarrow$ *i)a)* Let (x_n) be a Cauchy sequence. Assume it is not converging to some element $x \in C$. This means

cccc (2.1) $$\forall x \in C \exists \varepsilon(x) > 0 \forall n_0 \exists n > n_0 \, d(x_n, x) > \varepsilon .$$

Then

$$C \subset \bigcup_{x \in C} B(x, \frac{\varepsilon(x)}{2}) .$$

Let

$$C \subset B(y_1, \frac{\varepsilon(y_1)}{2}) \cup \cdots \cup B(y_1, \frac{\varepsilon(y_1)}{2})$$

be a finite subcover (compactness). Then there exists at least one $1 \le k \le m$ such that

$$A_k = \{n \in \mathbb{N} \mid d(x_n, y_k) < \frac{\varepsilon(y_k)}{2}\}$$

is infinite. Fix that k and apply the Cauchy criterion to find n_0 such that

$$d(x_n, x_{n'}) < \frac{\varepsilon(y_k)}{2}$$

for all $n, n' > n_0$. By (cccc 2.1), we can find an $n > n_0$ such that

$$d(x_n, y_k) > \varepsilon(y_k) .$$

Since A_k is infinite, we can find an $n' > n_0$ in A_k thus

$$\begin{aligned} \varepsilon(y_k) &< d(x_{n'}, y_k) \le d(x_n, x_{n'}) + d(x_{n'}, y_k) \\ &< \frac{\varepsilon(y_k)}{2} + \frac{\varepsilon(y_k)}{2} = \varepsilon(y_k) . \end{aligned}$$

A contradiction. Thus the Cauchy sequence has to converge to some point in C. ■

Corollary 2.7. *Every intervall* $[a, b] \subset \mathbb{R}$ *with* $a < b \in \mathbb{R}$ *is compact*

Proof: It is easy to see that $X \setminus [a,b]$ is open. Hence, by Proposition closed 2.1 $[a,b]$ is complete, i.e. i)a) is satisfied. Given $\varepsilon > 0$, we can find $k > \frac{1}{\varepsilon}$. For $m > k(b-a)$ we derive

$$[a,b] \subset \bigcup_{j=0}^{m} B(a + \frac{j}{k}, \varepsilon) \,.$$

Thus the Theorem main 2.6 applies. ■

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Lemma 2.8. *Let $r > 0$ and $n \in \mathbb{N}$, the set $C_r = [-r,r]^n$ is compact.*

Proof: Let $x \notin C_r$, then there exists an index $j \in \{1,..,n\}$ such that $|x_j| > r$. Let $\varepsilon = |x_j| - r$ and $y \in \mathbb{R}^n$ such that

$$\max_{i=1,..,n} |x_i - y_i| < \varepsilon \,,$$

then

$$|y_j| \;=\; |y_j - x_j + x_j| \;\geq\; |x_j| - |y_j - x_j| \;>\; |x_j| - \varepsilon \;=\; r \,.$$

thus $y \notin C_r$. Hence, C_r is closed and according to Proposition rncomp ??, we deduce that C_r is complete.
For $n = 1$ and $\varepsilon > 0$, we have seen above that for $k > \frac{1}{\varepsilon}$ and $m > \frac{2r}{k}$

$$[-r,r] \subset \bigcup_{j=0}^{m} B(-r + \frac{j}{k}, \varepsilon) \,.$$

Therefore

$$[-r,r]^n \subset \bigcup_{j_1,...,j_n=0,...m} B_\infty((-r + \frac{j_1}{k}, .., -r + \frac{j_n}{k}), \varepsilon) \,.$$

Thus i)a) and i)b) are satisfies and the Theorem main 2.6 implies the assertion (The separable dense subset is $\mathbb{Q}^n$.) ■

Theorem 2.9. *Let $C \subset \mathbb{R}^n$ be a subset. The following are equivalent*

1) *C is compact.*
2) *C is closed and there exists an r such that*

$$C \subset B(0,R) \,.$$

(That is C is bounded.)

Proof: 2) ⇒ 1) Let

$$C \subset B(0,R) \subset [-R,R]^n$$

be a closed set. Since $[-R,R]^n$ is compact, we deduce from Proposition subcom 2.4 that C is compact as well.

1) $\Rightarrow$ 2) Let C subset $\mathbb{R}^n$ be a compact set. According to Theorem main 2.6 i)b), we find

$$C \subset B(x_1, 1) \cup \cdots \cup B(x_m, 1)$$

thus for $r = \max_{i=1,..,m}(d(x_i, 0) + 1)$ we have

$$C \subset B(0, r) \ .$$

Moreover, by Theorem main 2.6 i)a) and Proposition closed 2.1, we deduce that C is closed. ∎

Proof: *ii)* : Let $x \in X$ and $0 < \varepsilon < 1$, then there exists a $\varepsilon_f > 0$ such that

$$d(x,y) < \varepsilon_f \quad \Rightarrow \quad d(f(x), f(y)) \; < \; \frac{\varepsilon}{3(1 + |g(x)|)} \; < \; 1 \, ,$$

and $\varepsilon_g > 0$ such that

$$d(x,y) < \varepsilon_g \quad \Rightarrow \quad d(g(x), g(y)) \; < \; \frac{\varepsilon}{2(1 + |g(x)|)} \; < \; 1 \, .$$

Let $y \in X$ such that $d(y,x) < \min\{\varepsilon_f, \varepsilon_g, 1\} = \delta$. Then we deduce form $\varepsilon < 1$

$$\begin{aligned} |fg(x) - fg(y)| &\leq |f(x)||g(x) - g(y)| + |f(x)g(y) - f(y)g(y)| \\ &\leq |f(x)||g(x) - g(y)| + |f(x) - f(y)||g(x)| + |f(x) - f(y)||g(x) - g(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon^2}{9} \; < \; \varepsilon \, . \end{aligned}$$