Tensor norms

Definition A tensor norm is an assignment α such that

1) For all pair of Banach space X, Y there is a norm α on a completion of $X \otimes Y$

$$X \otimes_\alpha Y = \overline{(X \otimes Y, \alpha)} \quad \text{completion}$$

ii) $\alpha(x \otimes y) = \|x\| \, \|y\|$

iii) $\forall S \; X_1 \to X_2 \; T \; Y_1 \to Y_2$ bounded

$$\| S \otimes T : X_1 \otimes_\alpha Y_1 \longrightarrow X_2 \otimes_\alpha Y_2 \| \le \|S\| \, \|T\|.$$

Definition 1)

$X \otimes_\varepsilon Y$ is the completion of $X \otimes Y$ with respect

$$\Big\| \sum_{j=1}^{m} x_j \otimes y_j \Big\|_\varepsilon = \sup_{\substack{\|x^*\| \le 1 \\ \|y^*\| \le 1}} \sum x^*(x_j) \, y^*(y_j)$$

2) $X \otimes_\pi Y$ is the completion of $X \otimes Y$ with respect to

$$\Big\| \sum_{j=1}^{n} z \Big\|_\pi = \inf_{z = \sum x_j \otimes y_j} \sum \|x_j\| \, \|y_j\|.$$

Remark

$$\|\underbrace{\sum x_j \otimes y_j}_{z}\|_\varepsilon = \|T_z : X^* \to Y^{**}\| = \|T_z^* : Y^* \to X^{**}\|$$

where $\quad T_z(x^*) = \sum x^*(x_j) y_j$

$\qquad T_z^*(y^*) = \sum x_j \, y^*(y_j)$

carefull : $\quad T_z : X^* \to \underline{Y}$

$\qquad T_z^* : Y^* \to X^{**}$ but has range in X.

Theorem $\otimes_\varepsilon$ is the smallest and $\otimes_\pi$ is the biggest tensor norm

Proof $\alpha(\sum x_j \otimes y_j) \leq \sum \alpha(x_j \otimes y_j) = \sum \|x_j\| \|y_j\|$

take the infimum. ✓

Claim $\alpha^*(x^* \otimes y^*) \leq \|x^*\|_{X^*} \|y^*\|_{Y^*}$

Indeed $\quad S\, X \to \mathbb{K} \qquad S(x) = x^*(x) \qquad \|S\| = \|x^*\|$

$\qquad T\, Y \to \mathbb{K} \qquad T(y) = y^*(y) \qquad \|T\| = \|y^*\|$

$$\alpha\left(\sum_k x^*(x_k) \otimes \sum y^*(y_k)\right) \leq \alpha\left(\sum x_k \otimes y_k\right)$$

$$\|$$

$$\left|\sum x^*(x_k)\, y^*(y_k)\right| \alpha(1 \otimes_1 1) = 1 \qquad \square$$