Lemma : $\Pi_p(T^{**}) = \Pi_p(T)$

Proof: Let x_1,...,x_m in X** and u(e_k)=x_k the corresponding linear map. Let y_1*,...,y_m^* be in the unit ball of Y* such that

T**(x_k)(y*_k)=||T**(x_k)||

By local reflexivity, we can find w:l_{p'}^m to X such that

||w|| $\leq (1+\varepsilon)\|u\|$

and |(w(e_k),T*(y*_k))| $\geq \frac{1}{1+\varepsilon}\|T^{**}(x_k)\|$

Hence

$$\left(\sum \|T^{**}(x_k)\|^p\right)^{1/p} \leq (1+\varepsilon)\left(\sum \|Tw(e_k)\|^p\right)^{1/p}$$

$$\leq (1+\varepsilon)^2 \Pi_p(T)\,\|u\|$$

Theorem: $\Pi_p^* = I_{p'}$

Proof: let $T: X \to Y$ finite rank, $T(X) \subset F$

Then

$$|tr(ST)| = |tr(S|_F T_{XF})| \leq \nu_{p'}(S|_X)\,\Pi_p(T^{**})$$

$$\leq I_{p'}(S)\,\Pi_p(T).$$

For the converse, we consider $\varphi: \mathcal{F}(X,Y)$

Let X_0, Y_0 be finite dimensional subspaces

Then $\|\varphi|_{\mathcal{F}(X_0,Y_0)}\| \leq \|\varphi\|$

Thus there exists $S\; Y_0 \to X_0$

$$v^0_{p'}(S\, X_0 \to Y_0) \le \|\varphi\|_{\Pi_p^*}$$

let $\underline{Y} \subseteq C(B_{Y^*})$ and $\hat{\alpha}: C(B_{Y^*})$

$$\begin{tikzcd}
\underline{Y} \arrow[r, phantom, "\supseteq"] & Y_0 \arrow[r] & X_0 \\
C(B_{Y^*}) \arrow[u, phantom, "\cap|"] \arrow[r, "\hat{\alpha}"'] & \ell_\infty^m \arrow[u, "\alpha"] \arrow[r, "D_\sigma"] & \ell_{p'}^m \arrow[u, "\beta"']
\end{tikzcd}$$

Then we define the w^*-limit

$$S: C(B_{Y^*}) \longrightarrow X^{**}$$

$$S(f)(x^*) = \lim_{Y_0 X_0} \langle \beta D_\sigma \hat{\alpha}(f), x^* \rangle$$

Then it is easy to see that S is p'-summing

and hence $\varphi(x^* \otimes y) = x^*(S(y))$ is p'-integral

$\blacksquare$

Proof (Lemma 1) We know that

$$\upsilon_p^0(X,Y)^\vee = \Pi_{p'}(Y, X^{**}) = \Pi_{p'}(Y,X)$$

Hence

$$(\upsilon_p^0)^{\vee\vee}(X,Y) = I_p(X, Y^{**})$$

In particular

$$\upsilon_p^0(T) = I_p(T : X \to Y^{**}) = \upsilon_p^0(T : X \to Y^{**})$$

$\square$