Trace duality and p integral maps

We have already seen the definition of the finite rank p nuclear maps which were designed such that

$$(\nu_p^o)^* = \Pi_{p'}$$

Now we want to calculate

$$\Pi_p^* = I_{p'}$$

The key here is the correction definition:

We say that T:X to Y is p integral if there exists a facorization

$$\begin{tikzcd} X \arrow[r] \arrow[d, "L"] & Y \arrow[r, hook] & Y^{**} \\ L_\infty(\mu) \arrow[r, "I_p"'] & L_p(\mu) \arrow[ur, "S"'] & \end{tikzcd}$$

$\mu(\Omega)=1$

I_p the natural inclusion map

$$I_p(T) = \inf \|L\| \, \|S\|$$

Recall : $p=2$ $\Pi_2 = I_2$

even without Y^{**} !

Lemma 1: Let X be finite dimensional. Then

$$v_p^0(T : X \to \underline{Y}) \leq I_p(T)$$

We need another preparation;

Proposition: Let u:X to L_∞ be a linear map and X a subspace of Y. The there exists v:Y to L_∞ extending u with the same norm.

$\|u\| \leq 1$ and

Proof: Let us first assume that X is finite dimensional and x_j, x_j* be an Auerbach basis. Given $\varepsilon > 0$ we may approximate u(x_j) by $\sum a_{kj} 1_{A_k}$ with m disjoint functions such that

$$\|u(x_j) - \underbrace{\sum a_{kj} 1_{A_k}}_{f_j}\|_\infty < \varepsilon$$

Then $w_\varepsilon = \sum x_j^* \otimes f_j$ satisfies

$$\|u - w_\varepsilon\| \leq \sum_{j=1}^n \|x_j^*\| \varepsilon < \varepsilon n$$

$$\|w_\varepsilon\| \leq \|u\| + \varepsilon n$$

Since $\mathrm{span}\{1_{A_k}\} \cong \ell_\infty^m$, we can extend w

to Y with $\|\hat{w}_\varepsilon\| = \|w_\varepsilon\|$

Then $w(y)(f) = \lim_{\varepsilon \to u} \int \hat{w}_\varepsilon(x) f \, d\mu$ is

a contraction with $w|_X = u$

Here $\lim_{\varepsilon \to \mathcal{U}}$ is a limit along an ultra filter containg all the sets $(0,\delta)$, $\delta > 0$.

For arbitrary X we consider
$\{F : F \subset X \; F \text{ finite dimensional}\}$
Then the weak*-limit

$$w(y) = \lim_{F \to \mathcal{U}} w_F(y)$$

yields a contractive extension of u,

provided $\mathcal{U}$ contains all the sets

$$I_F = \{F' : F' \supset F\} \quad \square$$

Proposition: Let X be finite dimensional, then

$$\upsilon_p^o(T : X \to Y^{**}) \le I_p^o(T).$$

Proof: Let $\varepsilon > 0$ and $u_\varepsilon X \to X_\varepsilon \subseteq \ell_\infty^m$ such that

$$\|u_\varepsilon\| \le 1 \quad \|u_\varepsilon^{-1}\| \le (1+\varepsilon)$$

We may also assume that

$$\begin{tikzcd}
& X_\varepsilon & X \arrow[l] \arrow[r] \arrow[d, "i"] & \underline{Y} \arrow[r, hook] & Y^{**} \\
\ell_\infty^m \arrow[ur, phantom, "\supseteq"] \arrow[rr, "w"] \arrow[d, "D_\sigma"] & & L_\infty \arrow[from=1-2, "\iota u_\varepsilon^{-1}"] \arrow[r, "I_p"] & L_p \arrow[ur] & \\
\ell_p^m \arrow[urrr, "R"] & & & &
\end{tikzcd}$$

Let $w: \ell_\infty^m \to L_\infty$ be an extension of ιu_ε^{-1}, $\|w\| \le (1+\varepsilon)$. Then $I_p w: \ell_\infty^m \to L_p$ is p-summing and hence we find $D_\sigma: \ell_\infty^m \to \ell_p^m$, a diagonal operator and $R: \ell_p^m \to L_p$ s.t.

$\|\sigma\|_p = 1$ $\quad \|R\| \le (1+\varepsilon)$

Define $x_k^* = u_\varepsilon^*(e_k^*)\sigma_k$ $\quad y_k = SR(e_k)$

Then $T = \sum x_k^* \otimes y_k \in Y^{**}$ and

$$\nu_p^\circ(T) \le \|\sigma\|_p \|SR\|$$
$$\le (1+\varepsilon) I_p(T) \quad \boxed{\varepsilon}$$

Proposition $\dim X < \infty$

$|\mathrm{tr}(TS)| \le \pi_p(T) I_{p'}(S)$

Proof $X \xrightarrow{T} Y \xrightarrow{S} X$

Let $T(X) \subset F$

$$I_{p'}(S|_F) \leq I_{p'}(S)$$

and by previous proposition

$$\upsilon^{0}_{p'}(S_F) \leq I_{p'}(S)$$

Thus $(\upsilon^{0}_{p'})' = \Pi_p$ implies

$$|\operatorname{tr}(TS)| \leq \Pi_p(T)\, \upsilon^{0}_{p'}(S) \leq \Pi_p(T)\, I_{p'}(S)$$

$\square$

Remark: For better results we need local reflexivity