Riesz transforms

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Alba Julia-Joint meeting AMS-Rumanian Society- June 2013
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joint work with Tao Mei and Javier Parcet
Riesz transforms in Harmonic Analysis and Operator Algebras

Plan:

- General semigroups and derivations
- Cocycles on groups and their derivations
- Possible use in operator algebras
- Riesz transform estimates
- A new perspective for classical (Fourier) multipliers
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$N$ is a (semi-) finite von Neumann algebra, with trace $\tau$;

$T_t = e^{-tA}$ is a family of completely positive unital trace preserving, and selfadjoint maps;

The gradient form (carre du champs) is defined as

$\Gamma(x, y) = A(x^* y) + x^* A(y) - A(x^* y)$.

Sauvageot-Cipriani: In general $\Gamma(x, x) \in \text{dom}(A_1/2)$.

In von Neumann algebras: One should require $\Gamma(x, x) \in L_1(N)$ for $x \in \text{dom}(A_1/2)$.

But that is not true for Sierpinski gasket!

(Meyer's problem) For which $A$ do we have $\|\Gamma(x, x)\|_1/2 \sim \|A_1/2 x\|_p$?

Motivation $\mathbb{R}^n$: Then $\nabla |\Delta|^{-1/2}$ is bounded.

Meyer showed Riesz transform estimate for infinite dimensional Ornstein-Uhlenbeck semigroup.
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ii) There exists $\xi$ in $M$ such that

$$\Gamma_{(I-T)}(x,y) = E([x,\xi]^*[y,\xi]).$$
Corollary

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*(Sauvageout-Cipriani/Davies Lindsay)* The $N \cap \text{dom}_2(A^{1/2})$ is a $^*$-algebra.

**Proof:** Let $\delta(x) = i[x, \xi]$ and $A = (I - T)$, then

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\| A^{1/2}(xy) \|_2 = \tau(\Gamma(xy, xy))^{1/2} = \tau(\delta(xy)^* \delta(xy))^{1/2}
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\]

The same inequality remain true if we passe to the limit:

\[
A = \lim_{\alpha} \frac{I - T_\alpha}{\alpha}.
\]

Rem.: Same remains true for \( N \cap \text{dom}_p(A^{1/2}) \) which contains \( T_t(N) \) by \( H^\infty \)-calculus.
Discrete groups

\[ N = \mathbb{L}(G) \]

with trace \[ \tau(\sum g a_g \lambda(g)) = a_1. \]

Special function \[ T(\lambda(g)) = e^{-t \psi(g)} \lambda(g) \]

where \( \psi \) is a conditionally negative function.

Recall (Schoenberg's theorem) that \( \psi \) is conditionally negative, i.e.
\[ \sum g a_g = 0 \Rightarrow \sum gh \bar{a}_g a_h \psi(g - 1 h) \leq 0 \]

if and only if every \( \phi_t \) is positive definite.

\( L^p(N, \tau) \) is the completion of \( N \) with respect to \[ \|x\|_p = \left[ \tau(|x|^p) \right]^{1/2} \]
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As part of Schoenberg’s proof one constructs a Hilbert space $H_\psi$ as the completion of the *real* Hilbert $\sum_g a_g \lambda(g) : \sum_g a_g = 0, a_g \in \mathbb{R}$ with inner product

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**Remark:** The free dilation from (JRS) will take place in $\Gamma_0(L_2(0, \infty) \otimes H_\varphi) \rtimes G$, where $\Gamma_0(H)$ is the von Neumann algebra generated by semicircular random variables $s(h)$ (Voiculescu’s functor).
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**Remark:** More familiar is the gaussian measure space construction $L_\infty(\Omega, \mu) \rtimes G$ generated by unitaries

$$\Gamma_1(H) = \{ e^{i B(h)} : B(h) \text{ centered, gaussian with variance } \|h\|^2 \}$$
Homomorphisms

For applications in harmonic analysis we use the family of group homomorphisms $\pi_t: G \to H \phi \rtimes \alpha$:

$$\pi_t(g) = (tb(g), \lambda(g)),$$

where $b(g) = (\delta e - \delta g) + N \psi \in H \phi$ is the canonical cocycle.

Let $T_{\Delta}t(\lambda(\xi)) = e^{-t\|\xi\|^2}$ be the heat semigroup on (the dual of) $\mathbb{R}^n$ (equipped with the discrete topology).

Then $T_{\Delta}t(\lambda(\pi_1(g))) = e^{-t\psi(g)}\lambda(\pi_1(g))$.

In other words: Semigroups for discrete groups are restrictions of classical heat semigroups to semidirect products.

Similarly $\rho(\lambda(g)) = e^{iB(b(g))}\lambda(g)$ is a trace preserving $\ast$-homomorphism from $L(G)$ to $\Gamma_1(H \phi \rtimes G)$.

Application of dilation technique (also Dabrowski): Let $G$ be a discrete weakly amenable group $G$ with Haagerup property, then $L(G)$ is strongly solid.
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Application of dilation technique (also Dabrowski): Let $G$ be a discrete weakly amenable group $G$ with Haagerup property, then $L(G)$ is strongly solid.
$G = \mathbb{R}^n$, \nabla (f) = (\frac{d}{dx}1, \ldots, \frac{d}{dx}n)(f);
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is the classical Riesz transform.

The gaussian measure space derivation is given by
$\delta(\lambda(g)) = B(b(g)) \lambda(g) \in \cap_{p<\infty} L^p(L^\infty(\Omega,\mu) \cdot G)

The Lebesgue space derivative $b(g)(x) = (b(g), x)$ is only locally integrable in $\mathbb{R}^n$ and ugly when considered with respect to the Haar measure of $\hat{\mathbb{R}}^n$ discrete.
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Riesz transforms

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Recall $L_2(LG) = \ell_2(G)$, $L_2(L_\infty(\Omega) \rtimes G) = L_2(\Omega) \otimes \ell_2(G)$. 

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\( \blacklozenge \) Lemma: \( xV - Vx = \delta(x)A^{-1/2} + V[x, A^{1/2}]A^{-1/2}. \)
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\( \blacklozenge \) Lemma: \( xV - Vx = \delta(x)A^{-1/2} + V[x, A^{1/2}]A^{-1/2} \). ‘In particular’, if \( \Gamma(x, x) \) and \( [A^{1/2}x] \) are bounded and \( A^{-1/2} \) is compact, then \( [V, x] \) is a compact operator.
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- Recall that $G$ has the Haagerup property if there exists conditionally negative $\psi$ such that $A_{\psi}^{-1}$ is compact on $\ell_2(G) = L_2(LG)$.
- Lemma: $xV - Vx = \delta(x)A^{-1/2} + V[x, A^{1/2}]A^{-1/2}$. ‘In particular’, if $\Gamma(x, x)$ and $[A^{1/2}x]$ are bounded and $A^{-1/2}$ is compact, then $[V, x]$ is a compact operator.
- Popa and Vaes use this argument for bi-exact groups to produce a separation of variables

$$\lambda(g)\ell_2(G)_{\rho(g)} \xrightarrow{V} \lambda(g)\rho(g)L_2(\Omega) \otimes \ell_2(G)_{\rho(g)} \xrightarrow{W} \lambda(g)L_2(\Omega) \otimes \ell_2(G)_{\rho(g)}.$$

Here $W$ is a classical fundamental unitary used in Fell absorption principle.
The argument seems reminiscent of Ozawa use of property AO:

\[
L(G) \otimes_{\min} R(G) \to B(\ell_2(G))/\mathcal{K}(\ell_2(G)) \\
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Theorem

For discrete groups and semigroups given by Herz-Schur multiplier as above, the Riesz transform is bounded from $L^p(LG) \to L^p(L^\infty(\Omega,\mu) \rtimes G)$ for $1 < p < \infty$.

Main ingredients:

i) Pisier's method, in particular the use of the tangent flow

$\alpha_t(f)(x,y) = f(x+ty,y)$

which is extended to $L^\infty(R^n \times R^n, d\mu \times d\gamma)$.

ii) A clever application of the Hilbert transform, and the Stein-Weiss transference method;

iii) Twisted Khintchine inequalities for $x = \sum h, g a h, g B(h) \lambda(g)$ which satisfy

$\|x\|_p \sim c\sqrt{p}\|E_{LG}(x^*x)\|_2^{1/2} + \|E_{LG}(xx^*)\|_p^{1/2}$.

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Examples:

Conditionally negative symmetric real functions on $\mathbb{R}^n$ are easy to characterize

$$\psi(\xi) = \int (1 - \cos((\xi, x))d\mu(x)$$

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Cocycle Hilbert space?
We have an explicit candidate for $H = L_2(\mu; \ell_2^2)$ and

$$b(\xi)(x) = \frac{1}{\sqrt{2}}(\pi_x(\xi)(e_1) - e_1) = \left(\begin{array}{c} \frac{\cos(\xi x) - 1}{\sqrt{2}} \\ \sin(\xi x) \end{array} \right).$$

The action is given by

$$\pi_x(\xi) = \left(\begin{array}{cc} \cos(\xi x) & -\sin(\xi x) \\ \sin(\xi x) & \cos(\xi x) \end{array} \right).$$
Multipliers

A multiplier is given by
\[ T_m(\lambda(g)) = m(g)\lambda(g). \]

For amenable groups \( T_m : L(G) \to L(G) \) is bounded if \( \hat{m} = \sum_g m(g)\lambda(g) \) is in \( L^1(L(G)) \).

Non-amenable groups (such as free groups) have a richer structure of completely bounded multipliers coming from representation theory.

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In classical analysis embedding theorems of Besov spaces $B_{\alpha,1} \subset L_1$ produce easy examples of multipliers.
Patch work multipliers

Bump function:

\[ \psi_d(z) = 4 \left( \delta^d z \right) \]

s.t. \[ \sum_{d \in \mathbb{Z}} \psi_d(z) = 1 \]

\[ m = \sum_{d} \psi_d m_d \]

\[ m_1, m_2, m_3, m_4 \]
Remark: Just assuming $m_j = \hat{f}_j$, $\sup_j \|f_j\|_1$ not enough. But maybe for norms $\|f_j\|_1 \leq \|f\|$?
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For a weight $w : \mathbb{R}^n \to [0, \infty)$, we define the Sobolev norm $\|f\|_{L^w_2} = (\int |\hat{f}(x)|^2 w(x) dx)^{1/2}$.
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Prop: Let $\varphi(\xi) = \int (1 - \cos(\xi x)) \frac{dx}{w(x)}$.
Let sup$_j \| g_j \|_{L^w_2} < \infty$. Then $m = \frac{1}{\sqrt{\varphi(\xi)}} \sum_j 1_l m g_j$ extends to a bounded linear map on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. 

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\sup_j \| g_j \|_{L^w_2} < \infty
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is enough for patching multipliers.
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Idea: Let \( G(x) = (0, g_j(x)) \). Then \( G \in H_\varphi \) and hence
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\langle G, V(\lambda(\xi)) \rangle = m(\xi)
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is an \( L_p \) multiplier.
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Let $\sup_j \|g_j\|_{L^w_2} < \infty$. Then $m = \frac{1}{\sqrt{\varphi(\xi)}} \sum_j 1_j m_{g_j}$ extends to a bounded linear map on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. If in addition $\int dx/w(x) < \infty$, then

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**Idea:** Let $G(x) = (0, g_j(x))$. Then $G \in H_\varphi$ and hence

$$\langle G, V(\lambda(\xi)) \rangle = m(\xi)$$

is an $L_p$ multiplier. The gluing part is a typical application of Littlewood-Paley theory in combination with noncommutative Khintchine inequalities.
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\sum_{k \leq j} 2^j \| \hat{\psi}_k * F_j \|_2^2 + \sum_{k \geq j} (1 + 2^k (k - j)^2) \| \hat{\psi}_k * F_j \|_2 < C
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Then $T_m$ is bounded on $L^p(\mathbb{R})$ for all $1 < p < \infty$, but in general not on $L^\infty$. 
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Some ‘harder multiplier results’ can be deduced from a clever application of cocycles, Pisier’s method, and ultimately, a transferred version of the Hilbert transform $m(\xi) = \text{sgn}(\xi)$ (with leads to the ‘only’ singular kernel).
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Thanks for listening