

Definition: A normed operator ideal is an assignment such that

1) For every pair of Banach spaces X , and Y , $\alpha(X, Y)$ a normed Banach space which contains the finite rank operators

$$2) \alpha(x^* \otimes y) = \|x^*\| \|y\| \text{ for all } x^* \in X^*, y \in Y$$

3) For all $S: X_1 \rightarrow X_2, R: Y \rightarrow Y_1$

$$\alpha(RS) \leq \|R\| \alpha(S)$$

Theorem: $L(X, Y)$ is the largest and $N(X, Y)$ is the smallest normed operator ideal.

Proof: We have to show that

$$\|T\| \leq \alpha(T) \leq N(T)$$

for all T . Indeed, let x in X and y^* in Y^* . Note that $\alpha(K, K) = K$. Then we define $S: K \rightarrow X$ by $S(t) = tx$ and $R(y) = y^*(y)$ from Y to K . It follows that

$$|y^*(Tx)| \leq \alpha(RS) \leq \|R\| \alpha(T) \|S\| = \|y^*\| \|x\| \alpha(T)$$

Let us recall that

$$N(T) = \inf \left\{ \sum \|x_k^*\| \|y_k\| \mid T(x) = \sum x_k^*(x) y_k \right\}$$

Using a Cauchy sequence argument, we deduce $\alpha(T) \leq N(T)$ from ii and the triangle inequality.

Remark: A tensor norm α on the couples (X^*, Y) allows us to define an ideal norm via,

$$\alpha(T) = \alpha\left(\sum x_k^* \otimes y_k\right) \quad T = \sum x_k^* \otimes y_k$$

Conversely, the restriction of an ideal norm to couples (X^*, Y) is a tensor norm (with restricted domain).

Duality: For an ideal norm we may define

$$\alpha^*(S) = \sup \{ |\text{tr}(ST)| : \alpha(T) \leq 1 \}$$

Here the supremum is taken over finite rank maps.

Remark: 1) This is not the only way to define a dual operation.

2) With this assignement

$$\alpha(X^*, Y)^* = \alpha(Y, X) .$$