For a given family F of finite dimensional spaces, we define

$$\gamma_p^{inj} = \gamma^o_{p,S(F)}$$

$$\gamma_p^{surj} = \gamma^o_{p,Q(F)}$$

$$\gamma_p^{inj,surj} = \gamma^o_{p,QS(F)},$$

where S(F) is the set of subspaces of elements in F, Q(F) the set of quotients of elements in F, and QS(F) the set o quotients and subspaces of elements in F.

Theorem: Let T:X to Y be finite rank. Then

1) gamma_{p,F}^o(T)<= 1 iff for all v: X*\to E_p, w:Y\to E_{p'} and S:E_p\to E_{p'}^* we have
    |tr(S*vT**u*)| <= sup_{Fin F} ||S\ten F^*:E_p(F*)\to E_{p'}(F)*||

2) gamma_{p,S(F)}^o(T)<= 1 iff for all v: X*\to L_p, w:Y\to E_{p'} and S:L_p\to E_{p'}^* we have
    |tr(S*vT**u*)| <= sup_{Fin F} ||S\ten F^*:L_p(F*)\to E_{p'}(F)*||

3) gamma_{p,Q(F)}^o(T)<= 1 iff for all v: X*\to E_p, w:Y\to E_{p'} and S:E_p\to L_{p'}^* we have
    |tr(S*vT**u*)| <= sup_{Fin F} ||S\ten F^*:E_p(F*)\to L_{p'}(F)*||

4) gamma_{p,QS(F)}^o(T)<= 1 iff for all v: X*\to L_p, w:Y\to L_{p'} and S:E_p\to E_{p'}^* we have
    |tr(S*vT**u*)| <= sup_{Fin F} ||S\ten F^*:E_p(F*)\to E_{p'}(F)*||

Proof: The first result follows from our factorization argument presented before. The 2) follows from 1) because we

$$\|S\ten id_Q: E_p(Q)\to Q(E_{p'}*)\|$$

for all quotients of spaces of l_p sums of F*'s. This allows us to apply the extension Lemma. The arguments in all the other cases are similar, using similar extension Lemma's. qed

Cor: For X and Y finite dimensional we may translate that to

$$\|R\ten T:E_p(X)\to E_{p'}(F^*)*\| <= \sup \|R\ten id_F\|$$

in case 1) or

$$\|R\ten T:E_p(X)\to L_{p}(F) \| <= \sup \|R\ten id_F\|$$

in case 2)

$$\|R\ten T:L_p(X)\to Q_{p}(F) \| <= \sup \|R\ten id_F\|$$

in case 3)

$$\|R\ten T: L_p(X)\to L_p(Y)\|<= \|R\ten id_F\|$$

where we consider finite dimensional spaces, and their subspaces and quotients.

Cor: An operator T \to Y admits a factorization T=vw. w:X\to H, v:H\to Y if and only if there exists a constant C such that
||a\ten T:l_2^n(X)\to l_2^n(Y)|| <= C \|a:\ell_2^n\t \ell_2^n||

The proof consists of two steps. For every finite dimensional subspace E of X and every finite dimensional quotient Q of Y, we see that T_{EQ}:E\to Q also satisfies the second inequality. We apply 4) in the previous case and find that
T_EQ = b*a
where a:E to l_2^m is a contraction, and b:Q*to l_2^m satisfies ||b||<(1+d) C. Then we use an ultraproduct
A= (a_EQd): X to prod_{(EQd)} l_2^m
and
B= (b_{EQd}) Y* to prod_{(EQd)} l_2^m, of norm <= C.
Then B*A coincides with i_YT:X to Y**, the composition of T with the inclusion map in the double dual.

Since an ultraproduct of Hilbert spaces is a Hilbert space, we conclude with H=A(X) and b B*|_{H}:A(X) to Y, the restriction to the image.

Similarly, we can prove: Let T:X to Y and I_Y:Y to Y** the inclusion map. Then i_YT factors through an L_p space if and only if
||R\ten T:E_p(X)\to Q_p(Y)||<= C ||R:E_p\to Q_p||
for some constant C>0 and all R:E_p \to Q_p from a finite dimensional subspace of l_p^n to a quotient of l_p^n.

Similarly, we see that i_YT factors though a subspace o L_p iff
||R\ten T:E_p(X)\to l_p^n(Y)||<= C ||R:E_p\to l_p^n ||
for some constant C>0 and all R:E_p \to l_p^n from a finite dimensional subspace of l_p^n to l_p^n.

Again we may formulate such a inequality for quotients and subspaces of quotients.

Remark: In the previous results we combined the abstract ultraproduct factorization results with with Kakutani's theorem: The class of ultraproducts is closed under taking ultraproducts.