<u>Theorem</u> (local reflexivity) Let X Banach space

$E \subset X^{**}$ and $F \subset X^*$, and $\varepsilon > 0$

Then there exists a map $u_\varepsilon : E \to X$ such that

$(1-\varepsilon)\|x\| \leq \|u_\varepsilon(x)\| \leq (1+\varepsilon)\|x\|$ and

$$f(u_\varepsilon(x)) = x(f) \qquad \forall\, x \in E,\ f \in F.$$

<u>Lemma</u> Let $\dim E < \infty$

$$L(E, X^{**}) = L(E, X)^{**}$$

Proof $L(E, X) = E^* \otimes_\varepsilon X$

We know that there is an isomorphism $u_\delta : E^* \to Y_\delta \subseteq \ell_\infty^m$

such that $u_\delta\, E^* \to Y_\delta \xrightarrow{u_\delta^{-1}} E^*$

$\|u_\delta\| \leq 1 \quad \|u_\delta^{-1}\| \leq 1+\delta.$

$$E^* \otimes_\varepsilon X \longrightarrow Y_\delta \otimes_\varepsilon X \subseteq \ell_\infty^m(X)$$

$$\Rightarrow (E^* \otimes_\varepsilon X)^{**} \longrightarrow (Y_\delta \otimes_\varepsilon X)^{**} \subseteq (\ell_\infty^m(X))^{**} = \ell_\infty^m(X^{**})$$

$$\| \qquad \ell_\infty^m \otimes_\varepsilon X^{**}$$

Note that the range of this map is contained $Y_\delta \otimes_\varepsilon X^{**}$

Thus for:

$z \in \cancel{(E \otimes_\varepsilon X)^{**}}\ E \otimes_\varepsilon X^{**} \longrightarrow z_\delta \in Y_\delta \otimes_\varepsilon X^{**} = (Y_\delta \otimes X)^{**}$

~~we find $z_\delta \in Y_\delta \otimes_\varepsilon X^{**}$ $\|z_\delta\| \leq 1$~~

hence ~~$\|u_\delta^{-1}(z_\delta)\|_{E \otimes_\varepsilon X^{**}}$~~

$$\|(u_\delta^{-1} \otimes Id)^{**}(z_\delta)\|_{(E \otimes_\varepsilon X)^{**}} \leq \|u_\delta^{-1}\| \, \|z_\delta\|$$

$$\leq (1+\delta)\, \|z\|_{E \otimes_\varepsilon X^{**}}.$$

Since $\delta > 0$ is arbitrary, we have equality $\blacksquare$

Prop $E \subset X^{**}$ $F \subset X^{*}$ finit dims. Then there exist

$u_\varepsilon : E \to E_\varepsilon \subset X$ such that

$\|u_\varepsilon\| \, \|u_\varepsilon^{-1}\| < (1+\varepsilon)$

$|f(u_\varepsilon(x)) - x(f)| < \varepsilon \|x\| \|f\|$

Proof let Δ be a delta net in E^*

and Σ be a delta net in E

Every function $x^* \in E^*$ can be extended to

$x^{***} \in X^{***}$ such that $\|x^{***}\| \le \|x^*\| \le 1$

and $x^{***}|_{E^*} = x^*$

Since $B_{X^{*}}$ is $\sigma(X^{***}, X^{**})$ dense in $B_{X^{***}}$

one star

we can find $\Delta \subset B_{X^*}$ such that

$\forall x \in X$

$$(1-\delta)\|x\| \le \sup_{x^* \in \Delta} |\langle x, x^*\rangle| \le (1+\varepsilon) \sup_{x^* \in \Delta} |\langle x, x^*\rangle|$$

Claim $\|x\| \le \frac{(1+\varepsilon)}{1-\delta(2+\varepsilon)} \sup_{x^* \in \Delta} |\langle x, x^*\rangle|$

Indeed assume $\|x\| = 1$ $s \in X$ $\|x - s\| < \delta$

$x_1 = \frac{x - s}{\|x - s\|}$ $s_2 \in X$ $\|x_1 - s_2\| < \delta$

$x_3 = \frac{x_1 - s_2}{\|x_2 - s_2\|}$ $s_3 \in X$ $\|x_3 - s_3\| < \delta$

can be done shorter

Then $x = s_1 + \sum_{k=1}^{\infty} \alpha_k s_k$ with $|\alpha_k| \le \delta^k$

Hence $\sup_{x^* \in \Delta^*} |\langle x, x^*\rangle| \ge \frac{1-\delta}{1+\varepsilon} \|s_1\| - \sum_{k=1}^{\infty} \delta^k$

$$\ge \frac{1-\delta}{1+\varepsilon} - \delta = \frac{1-\delta-\delta(1+\varepsilon)}{1+\varepsilon} = \frac{1-\delta(2+\varepsilon)}{1+\varepsilon}$$

Let $\tilde{F}$ be the finite dimensional subspace generated by F and $\tilde{\Delta}$

Fix an Auerbach basis for $\tilde{F}$ $f_1 \dots f_m$ $f_1^{*} \dots f_m^{*}$

such that $f_j^{*}(f_i) = \delta_{ij}$

Since $L(E, X)^{**} = L(E, X^{**})^{**}$

and $L E \to X^{**}$ (the inclusion map) has norm 1

We can find a map $u_\varepsilon \; E \to X$ such that

$$(*) \quad |\langle u_{\tilde{\varepsilon}}(x), f_j \rangle - \langle x, f_j \rangle| < \tilde{\varepsilon}\|x\| \text{ for all } j = 1 \dots \dim \tilde{F}$$

Then we ~~have for the map~~ compare them.

$$q_{\tilde{F}} u_\varepsilon \quad E \longrightarrow X \longrightarrow \tilde{F}^{*}$$

$$q_{\tilde{F}} L \quad E \hookrightarrow X^{**} \longrightarrow \tilde{F}^{*}$$

$$\|(q_{\tilde{F}} L)^{*}(f_j) - q_{\tilde{F}} u_\varepsilon (f_j)\| \leq \varepsilon$$

Hence $\displaystyle \| q_{\tilde{F}} u_{\tilde{\varepsilon}} - q_{\tilde{F}} L \| \leq \sum_{j=1}^{m} \|(q_{\tilde{F}} L)^{*}(f_j) - q_{\tilde{F}} u_\varepsilon)\|$

$< m\tilde{\varepsilon}$ Choose $\boxed{\tilde{\varepsilon} < \frac{\varepsilon}{m}}$

In particular

$$|\langle u_\varepsilon(x), f^*\rangle - \langle \cancel{u_\varepsilon} x, f^*\rangle| \le \tilde{\varepsilon}\, \|x\| \, \|f^*\| \qquad f^* \in F$$

Moreover

$$\|u_\varepsilon(\underline{x})\| \geq \sup_{f\in\Delta} |\langle u_\varepsilon(x), f\rangle|$$

$$= \sup_{f\in\Delta} |\langle ux, f\rangle| - \sup_{f\in\Delta} |\langle u_\varepsilon(x) - x, f\rangle|$$

$$\geq \frac{1-2\delta(1+\varepsilon)}{1+\varepsilon}\|x\| - \tilde{\varepsilon}\|x\|$$

$$\geq \left(\frac{1-2\delta(1+\varepsilon)}{1+\varepsilon} - \varepsilon\right)\|x\| \qquad \square$$

(How to do (a)) $\dim E < \infty$

Remark We also proved $\Delta \subset B_E$ δ net and

$w: E \to Y$ lin map

$$\|w\| \le \frac{1}{1-\delta} \sup_{s\in\Delta} \|w(s)\|$$

Indeed $x = \sum_{k=0}^{\infty} \alpha_k s_k \qquad |\alpha_k| \le \delta^k$

$$\|w(x)\| \le \sum_k |\alpha_k| \sup_{s\in\Delta} \|w(s)\|$$

$$\le \frac{1}{1-\delta} \sup_{s\in\Delta} \|w(s)\| \qquad \square$$

Rem The principle of local reflexivity as stated can be obtained from Helly's theorem (see Defant / Floret). However in practice our version is enough (personal experience)