Basic properties

Definition: A Banach space is given by a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, a norm $\| \cdot \| : X \to [0, \infty)$ satisfying:

1) $\| ax \| = |a| \| x \|$ for all $a \in \mathbb{K}$, $x \in X$

2) $\| x + y \| \leq \| x \| + \| y \|$

3) $\| x \| = 0 \iff x = 0$

4) Every Cauchy sequence $(x_n)$ is convergent.

(This means $\forall e > 0 \exists n_0 \forall n, m > n_0 \| x_n - x_m \| < e$)

Remark: a) $X$ satisfying (i), (ii), (iii) is called a normed space.

b) Every normed space can be completed to satisfy (iv).

c) (iii) is equivalent to

$$\sum \| x_n \| < \infty \implies \sum x_n \in X$$
Motivation: Historically the completeness property was crucial in finding solutions to differential and integral equations and hence came up as spaces of solutions.

Then Banach spaces got a life of their own.

Examples

\[(\mathbb{R}^n, \| \cdot \|_2) \quad \| x \|_2 = \left( \sum_{j=1}^{n} x_j^2 \right)^{1/2} \]

\[(\mathbb{R}^n, \| \cdot \|_p) \quad \| x \|_p = \left( \sum_{j=1}^{n} |x_j|^p \right)^{1/p} \]

\[C(\mathcal{X}) \quad \| f \| = \sup_{x \in \mathcal{X}} |f(x)| \quad \text{if compact} \]

\[C(\mathcal{X}, \| \cdot \|_2) \quad \| f \|_2 = \left( \int |f(x)|^2 \, d\mu(x) \right)^{1/2} \]

is not complete (even for \( L^2(0,1) \)).

This lead to Lebesque's integration theory and

\[L_p(\mathcal{X}, \mu) = \left\{ f : f \text{\, measurable} \mid \| f \|_p = \left( \int |f(x)|^p \, d\mu(x) \right)^{1/p} \right\} \]

**Note:** \( L_2([0,1]) \) is the completion of the named space mentioned above.
Contraction principles

Let \((X, \| \cdot \|)\) be a Banach space.

Then \(V \subseteq X\) be a closed subspace. Then \(V\) is a Banach space with induced norm

\[\| \cdot \|_V = \inf_{v \in V} \| x + v \|, \quad x \in X\]

Proof: \(V\) closed implies \(V\) closed.

Let \((x + V) = \{x + v : x \in X, v \in V\}\)

\[(x + V) + (y + V) = (x + y) + V\]

\[\|x + V\|_X = \inf_{v \in V} \|x + v\|\]

is a quotient Banach space of \(X\).

Note: Let \(V \subseteq X\) be a linear subspace.

Then \((V, \| \cdot \|)\) is a Banach space iff \(V\) is closed.

Note: We can always find a completion \(X \rightarrow X\).
Definition: \( X \) Banach, \( Y \) named. Then
\[
L(Y, X) = \{ T : Y \rightarrow X : T \text{ linear} \}
\]

\[
\|T\| = \sup_{\|y\| \leq 1} \|T(y)\|
\]

Is a Banach space

Let \( \sum T_n < \infty \). Define \( T(x) = \sum T_n(x) \)

This certainly converges and is linear

\[
\|T - \sum_{n=1}^{k} T_n(y)\| \\
\leq \sum_{n=1}^{k} \|T_n(y)\| \\
\leq \|T\| \sum_{n=k}^{\infty} \|T_n\| \\
\leq \|T\| \sum_{n=k}^{\infty} \|T_n\| \\
\rightarrow 0 \\
\text{as } k \rightarrow \infty
\]

Hence \( \|T - \sum_{n=1}^{k} T_n\| \leq \sum_{n=k}^{\infty} \|T_n\| \rightarrow 0 \)

and \( T \) is the norm limit

Application: \( Y^* = L(Y, K) \) is a Banach space
\( Y^{**} = (Y^*)^* \) is a Banach space.

Theorem: The map \( \psi : Y \rightarrow Y^{**} \) by \( \psi(y) = y^*(y) \)
(\#J) is an isometry.
Hahn–Banach in a nutshell

**V** vector space over \( \mathbb{R} \)

\( K \subseteq V \) convex \( \forall x, y \in K, 0 \leq \lambda \leq 1 \) \( \lambda x + (1-\lambda)y \in K \)

Assume \( 0 \in K \) \( P_0(x) = \sup \{ t : \frac{x}{t} \in K \} \)

Claim \( P_0(x+y) \leq P_0(x) + P_0(y) \)

Indeed \( \frac{x}{t_1}, \frac{y}{t_2} \in K \)

\[
\frac{x+y}{t_1 + t_2} = \frac{t_1}{t_1 + t_2} \frac{x}{t_1} + \frac{t_2}{t_1 + t_2} \frac{y}{t_2} \in K
\]

Hence \( P_0(x+y) \leq P_0(x) + P_0(y) \).

**Theorem:** \( p : W \to \mathbb{R} \) sublinear \( (p(x+y) \leq p(x) + p(y)) \)

\( p(tx) = tp(x) \quad t > 0 \)

\( \exists \ g : W \to \mathbb{R} \) linear \( \exists \ p \)

\[ \Rightarrow \exists T : V \to \mathbb{R} \quad \forall \ p \quad T|_W = p \]

**Lemma:** True for \( V = W + \mathbb{R} x_0 \) \( x_0 \notin V \)

\( \exists \ t > 0 \) \( \exists \) \( \alpha \) has to satisfy

\( F(0 + tx_0) = f(0) + t \alpha \)

\( f(0 + tx_0) = f(0) + t \alpha \)

\( t > 0 \) and \( t < 0 \)

\( f(0) - p(0 - tx_0) \leq t \alpha \)
two inequalities

\[ \alpha \leq \frac{p(w+t\xi) - f(w)}{t} \quad \forall w \in W, t > 0 \]

\[ \frac{f(w) - p(w-t\xi)}{t} \leq \alpha \quad \forall w \in W, t > 0 \]

Therefore it suffices to show that

\[ \frac{f(w') - p(w'-s\xi)}{s} \leq \frac{p(w+t\xi) - f(w)}{t} \]

\[ \Leftrightarrow tf(w') + sf(w) \leq tp(w'-s\xi) + sp(w+t\xi) \]

\[ \Leftrightarrow p(tw' - st\xi + sw + st\xi) \]

\[ \Leftrightarrow p(tw' + sw) \]

\[ \Leftrightarrow f(tw' + sw) \]

Theorem follows by Zorn's lemma, taking a maximal element \( w \in (\tilde{T}, \tilde{W}) \)

\[ f|_W = f \quad w \in \tilde{W} \]

\[ (\tilde{T}, \tilde{W}) \leq (\tilde{T}, \tilde{W}) \] if \( \tilde{W} \in \tilde{W} \)

\[ f|_W = f \]
Theorem \( K \subset V \) convex set \( \emptyset \neq K \)

Then there exists a linear functional \( \varphi : V \rightarrow \mathbb{R} \) such that

\[
\sup_{x \in K} \varphi(x) < \varphi(y)
\]

Proof: \( \varphi \) is sublinear.

\[
\begin{align*}
W &= \mathbb{R} y \\
\varphi(y) &= 1 \\
\varphi(tx) &= t \varphi(x) \\
\varphi(0) &= 0
\end{align*}
\]

\( \varphi : V \rightarrow \mathbb{R} \) a linear extension \( \text{We don't care about} \)

\[
\varphi(x) \leq \varphi(y)
\]

Then \( \varphi(x) \leq 1 \) for \( x \in K \)

Application 2 \( X \) normed space. Then there exist \( X^* \subset X^* \)

\[
\|x^*\| = 1 \quad |x^*(x)| = 1
\]

\( \text{Def: } D = \{1, 1\} \quad W = \mathbb{R} x \)

\( \varphi(x) = 1 \) \( x \in D \) extend
Applications

**Theorem**: Let $V$ be a normed vector space. Then there exists a compact set $K$ such that $V \subseteq C(K)$ isometrically.

**Lemma**: $K = \{ f : V \rightarrow \mathbb{K} : \|f\| \leq 1, f \text{ linear}, f(K) \text{ equipped with point wise topology} \}$ is compact.

**Proof**: Open sets: given $x_1, \ldots, x_n \in V$, $\epsilon > 0$

$$0 \in x_1, \ldots, x_n = \exists f : \|f(x_i)\| \leq \epsilon$$

**Special case**: $V$ is separable ($x_n$) dense

$$d(f,g) = \frac{\sum \|f(x_n) - g(x_n)\|}{\sum (1 + \|f(x_n) - g(x_n)\|)} \quad \frac{1}{2^n} \text{ is a metric}$$

We want to show $f^k$ has a convergent subsequence

a) $f^k(x_i)$ converges, find another subsequence such that $f^{d^2}(x_i)$ converges

$$d^2(x_i, x_j) = \|f^k(x_i) - f^k(x_j)\|$$

Then $f^{d^2}(x_i)$ converges for all $x_i$

It is easy to see that $f(x) = \lim_{k \to \infty} f^{d^2}(x)$ is a linear functional of norm $\leq 1$.

**Proof** (Thm): $i(V)(f) = f|V)$, $f \in K$ defined above is linear and, by H. B, $\|f\| = \sup_{f|x \in K} \|f(x)\| = \|i(V)f\|$ is isometric.

$W : (B_{\mathbb{K}}^\times, \sigma(B_{\mathbb{K}}^\times)(x))$ is compact by Tychonoff.
Finite dimensional spaces

Recall: all finite dimensional Banach spaces with fixed dimension "coincide" on all norms in $\mathbb{R}^n$ are equivalent.

We want to prove something more precise.

Theorem: Let $E$ be a $n$-dimensional Banach space. Then there exists vectors $x_1, \ldots, x_n \in E, x_1^*, \ldots, x_n^* \in E$ such that

$$x_i^*(x_k) = \delta_{ik} \quad \text{if} \quad x_j^* x_i^* = 1 = x_i^* x_i^*$$

Geometrically

$$(x_j, (x_j^*)^*)$$ is called Auerbach basis.
Determinants

A is an $n \times n$ matrix

$$\det A = \sum_{\sigma \in \text{Perm}_n} \text{sgn}(|\sigma|) \prod_{j=1}^{n} a_{\sigma(j)}$$  \hspace{1cm} (\star)$$

has the following properties:

$$\det(ab) = \det(a) \det(b)$$

$$\det(a) = 0 \iff a \text{ has not full rank}$$

$$\det(a) = \prod_{i=1}^{n} \lambda_i(a) \text{ if } a \text{ has eigenvalues}$$

$$\det(I) = 1$$

$$p(t) = \det(A + ta) \text{ satisfies}$$

$$p(0) = \text{tr}(a) = \sum_{i=1}^{n} a_{ii}$$

(Other properties not so easy to obtain from $\star$)

Def: Let $\alpha$ be a norm on $\text{Mat}_n = n \times n$ matrices

$$\alpha^*(a) = \sup_{\|b\| = 1} \|a(b)b^\top\| \quad \alpha(0) \leq 1$$

$$\sum_{k \in \mathbb{C}} \text{ for } \{e_i\} \text{ (upto } (1,1) \text{) } \to (\{e_i\} \text{ standard dual)}$$
Lemma (Leurs) Let $v$ be now on $H_n$. Then
there exist $a_0 \in H_n$ invertible such that
$$\alpha(a_0) = 1 \quad \alpha'(a_0) = n$$

**Proof:** $a_0$ such that
$$|\det(a_0)| = \sup \{|\det(a)| : \alpha(a) \leq \alpha_0 \}$$

Thus $|\det(a_0)| \geq \frac{1}{\alpha(n)^{\frac{1}{n}}} > 0$ hence invertible.

$$|\det(a_0 + tb)| \leq \alpha(a_0 + tb)^n |\det(a_0)|$$

$$\Rightarrow |\det(1 + t a_0' b)| \leq \alpha(1 + t b a_0 b)^n$$

$$= 1 + t \max\{b\} + o(t)$$

$$|1 + t b (a_0' b) + t^2| \quad t \in K$$

We may choose $t = \delta t \sqrt{n(b)}$ such that
$$\delta t \sqrt{n(b)} \cdot (a_0' b) > 0$$

Hence
$$1 + t |b(a_0' b)| + o(t^2) \leq 1 + \delta t \cdot n \cdot \delta t \cdot (a_0' b) + o(t^2)$$

Hence
$$|b(a_0' b)| \leq n \cdot \delta t \cdot (a_0' b) \quad \text{Hence} \quad \alpha'(a_0') \leq n.$$
Application 1. There exists an Auer back basis

\[ x_i = a(e_i) \text{ satisfy } \|x_i\| = 1 \]
and \( \alpha^{-1}(a^{-1}) = \mathbb{N} \)

What is dual norm

\[ \sup_{\|x\| \leq 1} \left| \sum_{i=1}^{m} b_i x_i \right| = \sup_{\|x\| \leq 1} \left| \sum_{i=1}^{m} \langle y_i, x_i \rangle \right| \]
where \( y_i \in E^* \) is given by

\[ \left( b_{i1}, \ldots, b_{in} \right) \]

Hence we find \( y_i \in E^* \)

\[ N = \sum_{i=1}^{m} \|y_i\| \|x_i\| = \frac{m}{2} \sum_{i=1}^{m} \langle y_i, x_i \rangle = \frac{m}{2} \|x_i\| \|x_i\| \in E \]

Hence \( \langle y_i, x_i \rangle = \|y_i\| \|x_i\| \in E \) for all \( i = 1 \ldots n \)
and \( \|x_i\| = 1 \) for all \( i = i = 1 \ldots n \).
However \( \langle y, x \rangle = s_y \) because we are working with a Hilbert space. Therefore

\[
\frac{\|y\|_x}{\|x\|} \geq \frac{\langle y, x \rangle}{\|x\|} = 1
\]

If \( \|x\|_x > 1 \) then \( \sum \|y\|_x > \|x\|_x \).

Thus \( \|y\|_x = 1 \) for all \( i = 1, \ldots, n \) \( \square \)

Recall complex case works analogously for Hahn–Banach.

Definition: A norm \( \| \cdot \|_x \) on \( V \) has enough symmetry if there exists a compact group of isometries such that

1. \( \alpha(g \cdot a g) = \alpha(a) \)
2. \( \int g \cdot a g \, d\mu(g) = \frac{\mu(a)}{n} \)

Lemma: \( \alpha(\text{id}) \alpha^*(\text{id}) = \| \cdot \|_x \)

Proof: Let \( b \in V \) such that \( \|b\| = \alpha^*(\text{id}) \). Then

\[
\alpha(g \cdot b g) \leq 1, \quad \|g \cdot b g\| \leq 1
\]

(see Banach) Thus

\[
\alpha(\int g \cdot b g) \leq 1
\]

\[
\alpha^*(\int g \cdot b g) \leq 1
\]
By property 1) we know that

\[ \int g^* b g = T_m(\alpha) \text{Id} \]

and \( \alpha (\int g^* b g ) \leq \alpha (\text{Id}) = T_m(\lambda) \\text{Id} = \lambda \text{Id} \)

\[ \int \alpha (g^* b g) = \alpha (\lambda) = \alpha (\text{Id}) \]

Therefore \( \lambda = T_m(\alpha) \) satisfies

\[ \alpha (\lambda \text{Id}) \leq 1 \]

\[ |\lambda| \alpha (\text{Id}) \]

\[ \frac{\alpha (\text{Id})}{n} \leq \alpha (\text{Id}) \leq \frac{\alpha (\text{Id})}{n} \]

\[ \frac{\lambda \alpha (\text{Id})}{n} = \frac{\lambda^* (\text{Id}) \alpha (\text{Id})}{n} \leq \frac{\lambda (\text{Id})}{n} \leq 1 \]

We have equality, and the claim follows. \[ \Box \]

Two reasons for existence of \( \alpha (\lambda) \alpha (\lambda^{-1}) = n \)

Question? Uniqueness?
Lemma: Assume such that

\[ x(au) = x(a) \quad \text{for all orthogonal matrices} \, \]

Then \[ x(a) x^*(a^t) = n \quad \text{and} \quad x(a) = 1 \]

\[ x(b) x^*(b^t) = n \quad \text{and} \quad x(b) = 1 \]

implies that \( b = au \) for some \( u \).

Proof: Wlog we may assume that \( a \) satisfies

\[ \sup \, |\det a| = |\det a|, \quad \text{and} \quad |\det a| \leq 1 \]

Consider \( C = b^{-1} a \) (invertible)

Using Gaveau-Schmidt we may assume

\[ C = \nabla \, \text{u} \quad \text{u orthogonal} \]

\[ \nabla \quad \text{upper diagonal} \]

Note that \( |\det C| = |\det b^{-1} a| = |\det b|^{-1} |\det b| \geq 1 \)

\[ \prod_{k=1}^{n} |\nabla_{kk}| \leq \left( \frac{\prod_{k=1}^{n} |\nabla_{kk}|}{n} \right)^n = \left( \frac{\prod_{k=1}^{n} (D_k \nabla^n)}{n} \right)^n \]

geometric mean = \[ \left( \frac{\prod_{k=1}^{n} (D_k \nabla^n)}{n} \right)^n \]

arithmetic mean = \[ \left( \frac{\prod_{k=1}^{n} (D_k \nabla^n)}{n} \right)^n \]
However

\[ |\text{tr} (\delta^* a \omega D_\nu) | \leq \alpha^*(\delta^* ) \alpha (a \omega D_\nu) \leq n \alpha (a) \leq n \]

Hence we have

\[ n \leq |\text{tr} (\delta^* a \omega D_\nu) | \leq \alpha^*(\delta^* ) \alpha (a) \leq n \]

\[ |K| = 0 \quad \text{Instead of Gram-Schmidt} \]

we use polar decomposition

\[ b^* a = w (F_d (|b^* a|)) u \]

and find

\[ |\det (b^* a) | = \left( \prod_d |d^* (|b^* a|) \right)^{1/n} > 1 \]

\[ \forall \]

\[ \frac{\sum_d |d^* (|b^* a|) |}{n} = \frac{|\text{tr} (b^* a \omega D_\nu) |}{n} \leq 1 \]

Equality here implies \( d^* (|b^* a|) = 1 \) hence

\( b^* a \) is already an unitary.

\[ |K| = \mathbb{R} \quad b^* a = \nabla u \quad \nabla \text{ is also complex matrix} \]

\[ \text{and we still have } 1 \leq |\det (\nabla u) | \leq \frac{1}{n} (|b^* a|) \leq 1 \]
We shall now show that \( \sqrt{(\mathbf{v} \cdot \mathbf{a})^2} \) is real and

\[
\| \mathbf{v} \cdot \mathbf{a} \| = \| \mathbf{v} \| \| \mathbf{a} \|
\]

Hence \( \mathbf{v} \cdot \mathbf{a} = \| \mathbf{v} \|^2 \mathbf{a} \) orthogonal.

Hence again from above shows that

\[ \mathbf{v} \cdot \mathbf{a} \] is unitary and hence \( \mathbf{v} \cdot \mathbf{a} \) is orthogonal.

**Interpretation:** The ellipsoid

\[
E = \mathbf{a} (B^m_2)
\]

\[ B^m_2 = \{ x \in \mathbb{R}^n : \| x \|_2 \leq 1 \} \]

is independent of the choice of \( \mathbf{a} \) with

\[
\alpha(a) = 1, \quad \omega(\alpha^\omega) = \nu
\]

**Application:** The ellipsoid \( E \), \( C \mathbb{B} = \{ x : \| x \|_E \leq 1 \} \)

of maximal volume satisfies

\[
\omega(\mathbb{U}^\omega) = \nu \quad \text{where} \quad \alpha = \| \mathbf{U} \|
\]

**Question:** What is \( \omega^2 \) ?

Same remark holds for ellipsoid of maximal volume containing \( \mathbb{B}_E \). Why ?
Some examples

\[ l^p = \left( \sum_{k=1}^{n} |x_k|^p \right)^{1/p} \]

\[ \|x\|_p = \|x\|_{l^2} \rightarrow l^p \| \]

Note \( \alpha \) is a right ideal norm.

\[ \alpha(\lambda x) = x, \|x\|_{l^2} \rightarrow \|x\|_1. \]

\[ G = \mathbb{Z}^n \times \text{Perm}_n \leq \text{U}(l^2^n) = \text{unitary group} \]

\[ \varepsilon \rightarrow D \varepsilon = \begin{pmatrix} \varepsilon_1 & \cdots & \varepsilon_n \end{pmatrix} \]

\[ \Pi \rightarrow M = (e_i) = \Pi_{\varepsilon(i)} \text{ permutation matrix} \]

Note \( D \varepsilon M \varepsilon D \varepsilon (0) = \varepsilon \varepsilon_i \varepsilon (0) \)

\[ = D \varepsilon \Pi M \varepsilon \]

\[ \varepsilon_i (g) = \varepsilon_g \varepsilon_i (g) \]

Hence \( \varepsilon D \varepsilon \varepsilon M \varepsilon \varepsilon D \varepsilon (i) \) is indeed a subgroup, and Haar measure = counting measure.
We need

\[ \alpha(g \cdot u g) = \alpha(u) \]

Indeed \( \Pi_u \) and \( D_u \) are isometries on \( E^n \)
and hence \( \alpha(g \cdot u g) = \alpha(u) \) (using left multiplications.)

Thus the ellipsoid of maximal volume in \( E^n \)
is given by

\[ B^m_2 \subset B^m_p \quad \text{for} \quad p > 2 \]

\[ \ell^m_p \subset B^m_p \quad \text{for} \quad 1 \leq p \leq 2 \]

Proof We know that \( \alpha(1d \cdot 1d \cdot 1d) = n \)

Thus it suffices to calculate

\[ \alpha(1d \cdot (1d - 1d_p) \cdot (1d)) = n (1d \cdot 2^n - 1d_p \cdot 1d) \]

For \( p > 2 \) we have \( (2(|x|_p)^p) \leq (2(|x|_2)^2 \)

and this is attained for \( x = (1,0,\ldots,0) \)

For \( 1 \leq p \leq 2 \) we have \( (2(|x|_p)^p) \leq n^{p-2} (2(|x|_2)^2 \)

and this is attained for \( x = (1,-1,-1) \)
Basis in infinite dimension

Let X be an infinite dimensional Banach space. A basis is a sequence \((x_n)\) such that

\[ x = \sum a_n x_n \]

for a unique choice of coefficients \((a_n)\).

In particular, the \((x_n)\)'s are linearly independent.

Ex 1: For the \(l_p\) spaces, the unit vector basis given by

\[ e_k = (0, \ldots, 0, 1, 0, \ldots) \]

form what is called an unconditional basis, i.e.

\[ \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leq C \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \]

Whenever \( \left\| \sum_{n=1}^{\infty} a_n x_n \right\| < \infty \)

**Thm:** \(l_1\) does not have an unconditional basis.

**Thm:** Let \((x_n)\) be a basis. Then there exists a constant \(K\) such that

\[ \left\| \sum_{n=1}^{m} a_n x_n \right\| \leq K \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \]

Whenever \( x = \sum a_n x_n \in X \).

Definition: \((x_n)\) is called a basic sequence, if \((x_n)\) is a basis for the closure of the set \( \{ \sum_{n=1}^{m} a_n x_n : m \in \mathbb{N} \} \).
We consider the space $X$ equipped with the new norm
\[ \|x\| = \sup_N \|P_N(x)\| \]
where $P_N(\sum_{n=1}^N x_n) = \sum_{n=1}^N x_n$ is the well-defined projection.

Note that $\|\cdot\|$ is finite by definition of the series.

Let $z_n$ be a Cauchy sequence. For every $N$, we may consider the finite dimensional space $X_N$ generated by the $x_1, \ldots, x_N$. Then $P_N(z_n)$ is also Cauchy and converges to some $z(N)$. For every $N < M$, the projection $P_{\{M,N\}}$ between finite linear maps is continuous, and hence $P_{\{M,N\}}(z_n)$ converges to $z(N) = P_{\{M,N\}}(z(M))$.

Therefore there exists a sequence $(a_k)$ such that
\[ z(N) = \sum_{k=1}^N a_k x_k. \]

We claim that $z(N)$ is a Cauchy sequence in $X$. Indeed, let $\varepsilon > 0$. Then we can find $k$ such that
\[ \|P_N(y^k) - P_N(y^{k'})\| < \frac{\varepsilon}{3}, \text{ uniformly in } n. \]

Choose $k_0 > k$ and $N$ such that
\[ \|P_N(y^k) - P_N(y^{k'})\| < \frac{\varepsilon}{3} \quad \text{for } N > N_0. \]

Then we get
\[ \|z(N) - z(M)\| \leq \|P_N(z) - P_M(z)\| + \|P_M(z) - P_N(z)\|. \]

Thus, we have $z(N)$ is Cauchy in $X$. Moreover,
\[ \|z(N) - P_M(z^k)\| = \lim_{d \to 0} \|P_M(z^d) - P_N(z^d)\| < \frac{\varepsilon}{3} \quad \text{for } k > k_0. \]

Now we show that $z(N)$ converges to $z$ in the $\|\cdot\|$ norm. Indeed, let $N > 0$. Then
\[ \|P_M(z) - P_M(z(N))\| = \|2(N - 2N)\| < \varepsilon \]
and for $M \leq N_0$, \[ \|P_M(z) - P_M(z(N))\| = 0. \]