Basis in infinite dimension

Let X be an infinite dimensional Banach space. A basis is a sequence (x_n) such that

$$x = \sum \alpha_n x_n$$

for a unique choice of coefficients (a_n).
In particular, the (x_n)'s are linearly independent.

Ex 1: For the l_p spaces, the unit vector basis given by

$$e_k = (0 \cdot \ \ \cdot 0, \underset{k\text{th position}}{1}, 0 \cdots)$$

form what is called an unconditional basis, i.e.

$$\left\| \sum_n \varepsilon_n \alpha_n x_n \right\| \leq C \left\| \sum_n \alpha_n x_n \right\|$$

whenever $\left\| \sum \alpha_n x_n \right\| < \infty$

Thm: L_1 does not have an unconditional basis.

Thm: Let (x_n) be a basis. Then there exists a constant K such that

$$\left\| \sum_{n=1}^{m} \alpha_n x_n \right\| \leq K \left\| \sum_n \alpha_n x_n \right\|$$

whenever $x = \sum \alpha_n x_n \in X$.

Definition: (x_n) is called a basic sequence, if (x_n) is a basis for the closure of the set $\{ \sum_{n=1}^{m} \alpha_n x_n : m \in \mathbb{N} \}$.

Proof:

We consider the space X equipped with the new norm

$$|||x||| = \sup_N \|P_N x\|_X,$$

where $P_N(\sum_n \alpha_n x_n) = \sum_{n=1}^{N} \alpha_n x_n$ is the well-defined projection.

Note that $|||\cdot|||$ is finite by definition of the series.

Let z_n be a Cauchy sequence. For every N, we may consider the finite dimensional space X_N generated by the x_1,....,x_N. Then P_N(z_n) is also Cauchy and converges to some z(N). For every N< M, the projection P_{M,N} between finite linear maps is continuous, and hence P_{M,N}(z_n) converges to z(N)=P_{M,N}(z(M)).

Therefore there exists a sequence (a_k) such that

$$z(N)= \sum_{k=1}^{N} a_k x_k .$$

We claim that z(N) is a Cauchy sequence in X. Indeed, let $\varepsilon > 0$

Then we can find k such that $|||y^k - y^l||| < \varepsilon/3$, hence

$$\|P_N(y^k) - P_N(y^l)\| < \varepsilon/3 \quad \text{uniformly in.}$$

Choose $k_0 > k_0$ and N such that

$$\|P_N(y^k) - P_M(y^k)\| < \varepsilon/3 \quad \text{for } N > N_0$$

Then we get

$$\|z(N) - z(M)\| \leq \|P_N(z^k) - z(N)\| + \|P_N(z^k) - P_M(z^k)\| + \|P_M(z^k) - z_M\| < \varepsilon$$

Hence $z(N)$ is Cauchy in X. Moreover,

$$\|z(M) - P_M(y^k)\| = \lim_j \|P_M(z^j) - P_N(y^k)\| \leq \varepsilon/3 \quad \text{for } k > k_0.$$

Now we show that $z(N)$ converges to M in the $|||\cdot|||$ norm.

Indeed, let $M > N$. Then

$$\|P_M(z) - P_M(z(N))\| = \|z_M - z_N\| < \varepsilon$$

and for $M \leq N_0$: $\|P_M(z) - P_M(z(N))\| = 0$

Therefore, both (X,|| ||) and (X, ||| |||) are complete normed vector spaces, and the inclusion (X,||| |||) $\subseteq$ (X,|| ||) is continuous. The identity map from (X,|| ||) to (X,||| |||) has a closed graph. Recall that T:X to Y has closed graph if

lim_n (x_n, T(x_n)) =(x,y)

implies T(x)=y. In our situation x_n has to converge in the usual and triple norm, and then, of cours,e y=lim_n y_n=x by the continuity of the inclusion. The closed graph theorem implies that the identity map is continuous, and this how we find our constant C.

Theorem: Let X and Y be Banach space and T: X to Y be a closed operator. Then T is continuous.

This result is a consequence of the open mapping theorem:

Theorem: Let T: X to Y be a continuous linear map between Banach spaces which is surjective, then the image of an open ball has non-empty interior.

For the proof we note that $\bigcup_n \overline{T(nB_X)} = Y$, and hence one of these sets has non-empty interior by the Baire category theorem. It is easy to conclude from here.

In order to prove the closed graph from open mapping theorem, consider the projection map P:G(T)={(x,Tx):x in X} to X. P is surjective by assumption, and the graph is a Banach space. Then B_X is contained in nB_{G(T)}, and hence ||x|| less than 1 implies ||Tx|| less than n.

Theorem:

The Fourier series form a basis in L_p for $1<p<\infty$
But not for $p=1$ and $p=\infty$.

Outline:

We first define the Poisson semigroup

$$P_t\left(\sum_k \alpha_k e^{ik\cdot}\right) = \sum_k \alpha_k e^{-t|k|} e^{ik\cdot}$$

and the bmo-norm

$$\|f\|_{bmo} = \sup_t \|P_t(|f|^2) - |P_t(f)|^2\|^{1/2}$$

Step I: $T: L_\infty \to$ bm bd and $T: L_2 \to L_2$ bd, then

$T: L_p \to L_p$ bd for $2<p<\infty$.

(without proof)

Step II: Apply this to $T\left(\sum \alpha_k e^{ik}\right) = \sum \operatorname{sgn}(k)\alpha_k e^{ik\cdot}$

(Exercise)

Step III $P\left(\sum_k \alpha_k e^{ik}\right) = \sum_{k\geq 0} \alpha_k e^{ik}$

is bounded on L_p, because

$$P = \frac{T + Id}{2}$$

Step III By Shifting

$$P_n(f) = e^{in\cdot} P(e^{-in\cdot} f)$$

is the projection onto $\sum_{k\geq n} \alpha_k e_{k}$; $I - P_n$ is the projection onto $\sum_{k\leq n} \alpha_k e^{ik\cdot}$

Then $\left\|\sum_{k=-n}^{n+1} \alpha_k e^{ik}\right\| \leq C_p \left\|\sum_k \alpha_k e^{ik}\right\|$ uniformly in n. ∎

Note: projection + linear independence, gives basis.

Showing that we do not have a basis for $p=\infty$, and

$C(K)$, uses the basis projection theorem: If it were a basis then

$$P_m(f) = \sum_{k=-n}^{n} \hat{f}(k) e^{ik\cdot}$$

would satisfy

$$\sup_m \|P_m : C(\mathbb{T}) \to C(\mathbb{T})\| \leq C$$

Using an elementary estimate, one can show that

$$\|P_m\| \geq c \log m$$

Hence no basis in $C(K)$, and L_1 (duality)

more: Exercises.

Theorem: The Haar basis is unconditional for $1<p<\infty$, but not for $p=\infty$.

It is a basis for $p=1$.

Idea: Probabilistic techniques (martingales) allow to show the required norm estimates.