Banach spaces-Homework 3

Due date: Friday, February 15

i) Let X be an n dimensional space. By fixing a basis $b_1, ..., b_n$, we define a norm α on $M_n(\mathbb{C})$ by

$$\alpha(a) \;=\; \|v_a : X \to \ell_2^n\| \;,$$

where $v_a(g_i) = \sum_j a_{ij} e_j$. Show that the Lewis map satisfying $\alpha(a) = 1$ and $\alpha^*(a^{-1}) = n$ gives the unique ellipsoid of minimal volume containing the unit ball B_X.

ii) Let $r_n(t) = \mathrm{sgn}(\sin(2\pi n t))$. Show that the closed linear span X of the family (r_n) has an unconditional basis in $L_p[0,1]$ for $1 \le p \le \infty$.

iii) Let $T : X \to Y$ and the dual operator $T^* : Y^* \to X^*$ be defined by $T^*(y^*) = y^* \circ T$.

(a) Show that $\|T\| = \|T^*\|$.

(b) Recall that T is compact if $T(B_X)$ is relatively compact. Show that T^* is compact iff T is compact. (Of course you should dig the literature and indicate the main idea in your solution.)

iv) (a) Find a proof of the open mapping theorem, and indicate why

$$Y \;=\; \bigcup_n \overline{T(nB_X)}$$

implies that for some $n \in \mathbb{N}$, the image of $T(nB_Y^\circ)$ of the open ball contains an open point.

(b) Use the closed graph theorem to show that for $p < q$ the identity map from ℓ_p to ℓ_q can not be surjective. What is left to prove?