Banach spaces-Homework 2

Due Date: February 1

1. Let X, Y be Banach space and Z be a Banach space with continuous injective maps $\iota_X : X \to Z$ and $\iota_Y : Y \to Z$. Let

$$X + Y \;=\; \{\iota_X(x) + \iota_Y(y) : x \in X, y \in Y\}$$

equipped with the norm

$$\|z\|_{K_t} \;=\; \inf_{z=\iota_X(x)+\iota_Y(y)} \|x\| + t\|y\| \;.$$

Show that $(X + Y, K_t)$ is a Banach space. Can you describe the dual space? Let $X = (\mathbb{K}^n, \| \|_1)$ and $Y = Z = (\mathbb{K}^n, \| \|_\infty)$. Calculate

$$\|(1, 1, ..., 1)\|_{K_t} \quad \|(1, ..., 1)\|_{K_t^*}$$

for all $t > 0$.

2. Let α be a norm on $M_n(\mathbb{R})$ such that

$$\alpha(D_\varepsilon a D_\varepsilon) \;=\; \alpha(a)$$

holds for all diagonal matrices D_ε with entries $(\varepsilon_1, ..., \varepsilon_n)$ and $\varepsilon_j = \pm 1$, and all a. Show that there exists an invertible diagonal matrix D_λ with

$$\alpha(D_\lambda)\alpha^*(D_\lambda^{-1} \;=\; n \;.$$

3. Let X be a norm on $\mathbb{R}^n$ such that

$$\| \sum_{i=1}^{n} \varepsilon_i \alpha_i e_i \|_X \;=\; \| \sum_{i=1}^{n} \alpha_i e_i \|$$

for all $\varepsilon_i = \pm 1$. Show that there exists a diagonal map D_λ such that

$$\|\lambda_i e_i\|_X \;=\; 1 \;=\; \|\lambda_i^{-1} e_i\|_{X^*} \;.$$

Here e_i are the standard unit vectors.