Banach spaces-Homework 2

Due Date: February 1

1. Let $X, Y$ be Banach space and $Z$ be a Banach space with continuous injective maps $\iota_X : X \to Z$ and $\iota_Y : Y \to Z$. Let

$$X + Y = \{ \iota_X(x) + \iota_Y(y) : x \in X, y \in Y \}$$

equipped with the norm

$$\|z\|_{K_t} = \inf_{z = \iota_X(x) + \iota_Y(y)} \|x\| + t \|y\| .$$

Show that $(X + Y, K_t)$ is a Banach space. Can you describe the dual space?

Let $X = (\mathbb{K}^n, \| \cdot \|_1)$ and $Y = Z = (\mathbb{K}^n, \| \cdot \|_\infty)$. Calculate

$$\|(1,1,\ldots,1)\|_{K_t}, \|(1,\ldots,1)\|_{K_t^*}$$

for all $t > 0$.

2. Let $\alpha$ be a norm on $M_n(\mathbb{R})$ such that

$$\alpha(D_\varepsilon a D_\varepsilon) = \alpha(a)$$

holds for all diagonal matrices $D_\varepsilon$ with entries $(\varepsilon_1,\ldots,\varepsilon_n)$ and $\varepsilon_j = \pm 1$, and all $a$. Show that there exists an invertible diagonal matrix $D_\lambda$ with

$$\alpha(D_\lambda) \alpha^*(D_\lambda^{-1}) = n .$$

3. Let $X$ be a norm on $\mathbb{R}^n$ such that

$$\| \sum_{i=1}^n \varepsilon_i \alpha_i e_i \|_X = \| \sum_{i=1}^n \alpha_i e_i \|$$

for all $\varepsilon_i = \pm 1$. Show that there exists a diagonal map $D_\lambda$ such that

$$\| \lambda_i e_i \|_X = 1 = \| \lambda_i^{-1} e_i \|_{X^*} .$$

Here $e_i$ are the standard unit vectors.