Banach spaces-Homework 1

Due Date: January 25.

1. Show that every normed space X has a unique completion. The means there exists a Banach space B and an isometric embedding $\iota : X \to B$ such that $\iota(X)$ is dense. Here unique means that if there two such Banach spaces B_1, B_2 with embeddings ι_1, ι_2. Then there exists an isometric isomorphism $T : B_1 \to B_2$ such that $T(\iota_1(x)) = \iota_2(x)$. (Hint: Define a suitable norm on the space Y of equivalence classes of Cauchy sequences).

2. Let X be a Banach space. Let $J_p(X)$ be the space of all sequences such that

$$\|x\|_{J_p(X)} \;=\; \sup_{i_0<i_1<\cdots} \left(\|x_{i_0}\|^p + \sum_{j\,\geq\,1} \|x_{i_j} - x_{j-1}\|^p \right)^{\frac{1}{p}}$$

is finite. Show that $(J_p(X), \| \, \|_{J_p(X)})$ is Banach space. Also show that $J_1(X)$ and $\ell_1(X)$ are isometrically isomorphic.

3. Let $1 \leq p, q, \leq \infty$. We consider the Banach space

$$L_p \oplus_1 L_q \;=\; \{(x,y) : x \in L_p, y \in L_q\}$$

equipped with the norm

$$\|(x,y)\| \;=\; \|x\|_p + \|y\|_q \;.$$

Here x, y stand for equivalence classes of functions. Let $V = \{(x,x) : x \in L_p \text{and} x \in L_q\}$. Show that there is an injective continuous embedding linear map $u : L_p \oplus L_q/V \to L_0$. Recall here that L_0 is the space of measurable functions equipped with the metric

$$d_0([f],[g]) \;=\; \inf\{\varepsilon : \mu(|f-g| > \varepsilon) < \varepsilon\} \;.$$

Consider the measure space $(\mathbb{R}, \Sigma, \lambda)$, where λ is the Lebesgue measure an Σ the σ-algebra of measurable functions. Show that the range of u is strictly bigger than L_p and L_q. Why does that not work for $[0,1]$?