Banach lattices:

A Banach lattice is given by a partially ordered space X, which is also a real Banach space, such that

i) $x<y$ implies $x+z<y+z$

ii) The set $C=\{x>0\}$ is a convex cone

iii) for every pair (x,y) the exists a least upper bound xvy and and a minimum $x\&y=-((-x)\ v\ (-y))$.

iv) $|x|=xv(-x)$ satisfies

$$\| |x| \| = \| x \|.$$

Example: X is a functions space of real functions with values in R, such that the maximum of two functions is again in X, or equivalently the supremum and iv) is satisfied.

Remark: A Banach lattice is si-complete if every countable bounded family has a lower bound, lub.

Remark: This is true for dual spaces of Banach lattices. Here

$$x^*>0 \text{ if } x^*(x)>0 \text{ for all } x>0.$$

The one can define

$$\sup F(x)= \sup_{x^* \in F} x^*(x)$$

one the positive cone and then extend to a linear functional on X.

Every Banach lattices admits a si-complete extension, by embedding X in its bidual.

For si-complete Banach lattices one can always define

$$\left(\sum |x_i|^2\right)^{1/2} = \sup_{\sum|\alpha_i|^2 \leq 1} \sum_i \alpha_i x_i$$

For a si-complete Banach lattice and positive x, one can define

P_x(y)= sup_n (nx v y)

The family of operators P_x, x>0 are commuting and form a boolean algebra. Moreover two elements are disjoint if

|x| & |y| = 0

For disjoint elements x and y one can easily show that P_xP_y=0.

An element e is called an order unit if for all x>0 one has x <n e for some e.

For a fixed element e one can define sublattice

M(e)=span { P_x(e): x>0}

Definition: A Banach lattice is called abstract L_p space if

$$x, y \text{ disjoint} \quad \text{implies} \quad \|x+y\|^p=\|x\|^p+\|y\|^p \quad .$$

Theorem: (Kakutani) An abstract L_p space is an L_p space.

Proof (sketch:) We will only discuss $p< \infty$.

The one can show that X is order complete, and X is a l_p direct sum of irreducible spaces M(e). The measure is then defined by

$$\mu(x)=\|P_x(e)\|_X^p$$

Additivity for disjoint `sets' x follows immediately. By a result of Stone, the boulean lattice comes as clopen (open and closed sets) of a totally disconnected compact Hausdorff space. Then

$$V\left(\sum a_j P_{x_j}(e)\right)= \sum_j a_j 1_{x_j}$$

extends to a linear isometry.

Theorem: An ultraproduct of L_p spaces in an L_p space.

Proof: It suffices to observe that an ultraproduct of Banach lattices is a Banach lattice. Here we define (x_i)<(y_i) if there are representatives such that the set of all i with x_i<y_i is in the ultrafilter. It is trivially clear that the abstract L_p property is preserved.