II p-summing maps

Def: $w_p(x_k) = \sup_{\|x^*\| \le 1} \left(\sum_k |x^*(x_k)|^p\right)^{1/p}$

$u_{\vec{x}}(\alpha) = \sum \alpha_k x_k$ or $u_{\vec{x}}(e_k) = x_k$

Lemma $w_p(x_k) = \| u_{\vec{x}} \, \ell_{p'} \longrightarrow X \|$

Proof $\| u_{\vec{x}} \, \ell_{p'} \to X \| = \sup_{\sum |\alpha_k|^{p'} \le 1} \| \sum \alpha_k x_k \|_X$

$= \sup_{\sum |\alpha_k|^{p'} \le 1} \sup_{\|x^*\| \le 1} |\sum \alpha_k x^*(x_k)|$

$= \sup_{\|x^*\| \le 1} \left(\sum |x^*(x_k)|^p\right)^{1/p}$

$\blacksquare$

Q $u_{\vec{x}}$ well-defined? $p=1$ we need

$u_{\vec{x}} : c_0 \longrightarrow X$

$c_0 = \{(\alpha_k) : \lim_k \alpha_k = 0\}$

bc span $\{e_k\}$ norm dense (unique extension by continuity)

Def $T: X \to Y$ is called p-summing if there exists a constant $C > 0$ such that

$$\sum_k \|Tx_k\|^p \le C^p \sup_{\|x^*\| \le 1} \sum |x^*(x_k)|^p \quad (\ast)$$

for all finite sequences $(x_k) \subset X$

$$\pi_p(T) = \inf\{ C \mid (\ast) \}$$

$$\Pi_p(E, F) = \{ T \mid \pi_p(T) < \infty \}$$

Note Π_p is an ideal (HW)

<u>Theorem</u> $T: X \to Y$ is p-summing iff there exists a $\overbrace{\text{measure}}^{\text{Radon}}$ μ on B_{X^*} such that

$$\|Tx\| \le C \left(\int |x^*(x)|^p \, d\mu \right)^{1/p} \quad (1 \le p < \infty)$$

Moreover $C = \pi_p(T)$. (can be choosen)

<u>Proof</u> $\mathcal{C} = \{ f : \underbrace{(B_{X^*}, \sigma(X^*, X))}_{\text{weak}^*\text{-topology}} \to \mathbb{R} \mid \sup f < 0 \}$

$\mathcal{C}$ is convex cone (i.e convex and $r > 0$ $f \in \mathcal{C}$ $\Rightarrow rf \in \mathcal{C}$)

$$\mathcal{C}_+ = \{ f_{\vec{x}}(x^*) = \sum_k \left(|x^*(x_k)|^p - \pi_p(T)^{-p} \|Tx_k\|^p \right) \}$$

is also a cone. Indeed $\tilde{x}_k = \begin{cases} x_k & 1 \le k \le n \\ y_{k-n} & n < k \le n+m \end{cases}$

$$f_{\vec{x}}(x^*) + f_{\vec{y}}(x^*) = \sum_{k=1}^{n+m} |x^*(x_k)|^p - \pi_p(T)^{-p} \|Tx_k\|^p$$

and $r f_{\vec{x}}(x^*) = f_{r^{1/p}\vec{x}}(x^*) \quad \forall x^*$?

Moreover $\mathcal{C}_+ \cap \mathcal{C}_- = \emptyset$

By Hahn Banach there exists a real continuous (bc $\mathcal{C}_-$ open)

linear function $\ell : C(B_{X^*}) \longrightarrow \mathbb{R}$

$$\beta = \sup_{g \in \mathcal{C}_-} \ell(g) \le \inf_{g \in \mathcal{C}_+} \ell(g) = \alpha$$

Claim $\alpha \le 0$

If $\alpha > 0$ $\alpha < \ell(g) < 2\alpha$

$\Rightarrow r\alpha < \ell(rg) < r2\alpha \xrightarrow[r \to 0]{} 0$

hence $\ell(rg) < \alpha$ for some r ↯

$\beta \ge 0$ same argument.

Thus $\beta = \alpha = 0$

Claim $g \geq 0 \implies \ell(g) \geq 0$

Indeed $-g - \varepsilon \underline{1} \in \underline{C}$

$$\left.\begin{array}{l} \ell(-g-\varepsilon 1) \leq 0 \\ \ell(g) + \varepsilon \ell(1) \geq 0 \end{array}\right] \implies \ell(g) \geq 0$$

By Riesz Representation theorem, there exists a positive measure μ on $C(B_{X^*})$ such that

$$\int g \, d\mu \geq 0 \qquad \forall g \in C_+$$

Thus $$\int g \frac{d\mu}{\mu(B_{X^*})} \geq 0 \qquad \forall g \in C_+$$

Hence we may assume $\mu(B_{X^*}) = 1$

Therefore we get

$$\int |x^*(x)|^p - \pi_p(T)^{-p} \|Tx\|^p \, d\mu \geq 0$$

$$\implies \int |x^*(x)|^p \geq \pi_p(T)^{-p} \|Tx\|^p$$

Conversely, let $x_1 \ldots x_n \in X$ then

$$\sum \|Tx_i\|^p \leq C^p \int \sum |x^*(x_i)|^p \, d\mu \leq C^p \sup_{\|x^*\| \leq 1} \sum |x^*(x_i)|^p \quad \blacksquare$$

Remark: let $S \subset B_{X^*}$ be a closed subset such that still

$$\sup_{s \in S} |s(x)| = \|x\|$$

Then we may replace B_{X^*} by S (bc we don't change w_p – norm)

Example / Cor let $T : c_0 \to X$ be p-summ, then there exists $\mu_n \geq 0$, $\sum \mu_n = 1$ such that

$$\|Tx\|_X^p \leq \pi_p(T) \sum_k \mu_k |x_k|^p$$

Indeed, we may use $S = \mathbb{N}$ with the ~~wea~~ $\sigma(\ell_1, c_0)$ topology

$$\boxed{\begin{matrix} c_0 \subseteq C(B_{\ell_1}) \\ \quad \cup \\ \subseteq c_0(\mathbb{N}) \end{matrix}}$$

$K = \overline{\mathbb{N}}$ one point compactification

which we may ignore later

Gen rem $\boxed{\begin{matrix} X \xrightarrow{T} Y \\ \cap \\ C(K) \end{matrix}}$ $\Rightarrow$ $\exists \mu$ on K

$$\|Tx\|_p \leq \pi_p(T) \left(\int_K |f(x)(k)|^p \right)^{1/p}$$

trace duality in this context

Recall $X \to Y$ Banach spaces & norm

$$\alpha^*(S) = \sup\{ tr(TS) \mid S \text{ finite rank } \alpha(S) \leq 1 \}$$

is called the adjoint norm

(Rem: there are variations ...)

Here S $Y \to X$ finite rank, i.e.

$$S(y) = \sum_{j=1}^{m} y_j^*(y)\, x_j$$

Then $TS(y) = \sum y_j^*(y)\, Tx_j$

and we define

$$tr(TS) = \sum_{j=1}^{m} y_j^*(Tx_j)$$

well-defined?

Lemma $T: Y \to Y$ finite rank

$$\left.\begin{aligned} T(y) &= \sum y_j^*(y)\, y_j \\ T(y) &= \sum \tilde{y}_k^*(y)\, \tilde{y}_k \end{aligned}\right] \Rightarrow \sum y_j^*(y_j) = \sum \tilde{y}_k^*(\tilde{y}_k)$$

Proof Let $E = T(Y) \subseteq Y$ be the range of T

Let $E \subset F$ such that $y \in F$, $\tilde{y}_k \in F$

Let $z_1, \dots, z_m$ be a basis of F (Auerbach basis)
$z_1^\vee, \dots, z_m^\vee$ dual functionals

Claim $$T(y) = \sum_{j=1}^{m} z_j^\vee(T(y))\, z_j$$ obvious

Claim $$\sum z_j^\vee(T z_j) = \sum y_j^\vee(y_j)$$

Indeed $$T(y) = \sum_{j=1}^{m} y_j^\vee(y)\, y_j$$

$$= \sum_{j=1}^{m} y_j^\vee(y) \sum_{k=1}^{m} z_k^\vee(y_j)\, z_k$$

$$= \sum_{k=1}^{m} \left(\sum_j z_k^\vee(y_j)\, y_j^\vee(y) \right) z_k$$

This means $$z_k^\vee(T(y)) = \sum_j y_j^\vee(y)\, z_k^\vee(y_j) \quad \forall y \,!$$

Therefore $$\sum_k z_k^\vee(T(z_k)) = \sum_k \sum_j y_j^\vee(z_k)\, z_k^\vee(y_j)$$

$$= \sum_j \sum_k y_j^\vee(z_k)\, z_k^\vee(y_j)$$

$$\sum_{j=1}^{n} y_j^\vee(y_j) = \sum_{j=1}^{n} y_j^\vee\Big(\sum_k z_k^\vee(y_j)\, z_k\Big)$$

$$= \sum_{jk} z_k^\vee(y_j)\, y_j^\vee(z_k) = \sum_k T^\vee z_k^\vee(z_k)$$

The same holds for the $\sim$ and we are done. $\square$

<u>Pb</u> find α such that

$$\boxed{\alpha^*(S) = \pi_p(S)}$$

Motivation

$$\Big(\sum_1^n \|Tx_k\|_Y^p\Big)^{1/p} = \sup_{(\sum \|y_k^\vee\|^{p'})^{1/p'} \le 1} \Big|\sum \langle y_k^\vee, Tx_k\rangle\Big|$$

$$\begin{tikzcd}
X \arrow[r, "T"] & Y \arrow[d] & y \arrow[d, mapsto] \\
\ell_{p'}^m \arrow[u, "u_{\bar x} \hat{=} (x_k)"] \arrow[r, equal] & \ell_{p'}^m & (y_k^*(y))_{k=1}^n
\end{tikzcd}$$

Def $\nu_q^\circ(S : Y \to X) = \inf w_{q'}(x_k) \Big(\sum \|y_k^*\|^q\Big)^{1/q}$

$$S(y) = \sum_{k=1}^{n} y_k^\vee(y)\, x_k$$

<u>Lemma</u> ν_q° defines a norm on $\boxed{\mathcal{F}(Y, X) = \text{finite rank maps}}$

Proof

We need $ab = \inf_{t>0} \frac{(ta)^q}{q} + \frac{(t^{-1}b)^{q'}}{q'}$

$S_1 = \sum_{k=1}^{n} y_k^* \otimes x_k \quad = \quad \sum s y_k^* \otimes s^{-1} x_k$

$S_2 = \sum_{j=n+1}^{m} y_j^* \otimes x_j \quad = \quad \sum t y_j^* \otimes t^{-1} x_j$

convenience

where $\boxed{y^* \otimes x(y) = y^*(y)\, x}$

is the associated linear map

let $t>0$ be arbitrary
$s>0$

① $t = s = 1$

$S_1 + S_2(y) = \sum_{j=1}^{m} y_j^* \otimes x_j$

$v_q^\circ(S_1+S_2) \leq \left(\sum_{j=1}^{m} \|y_j^*\|^q\right)^{1/q} w_{q'}(x_1 \ldots x_m)$

$\cancel{\leq \left[\left(\sum_{j=1}^{n} \|y_j^*\|^q\right)^{q/q} + \left(\sum_{j=m+1}^{m} \|y_j\|^q\right)^{q/q}\right]^{1/q}}$

$\leq \frac{1}{q} \sum_{j=1}^{m} \|y_j^*\|^q + \frac{1}{q'} \sup_{x^*} \sum_{j=1}^{m} |x^*(x_j)|^{q'}$

$\leq \frac{1}{q} \sum_{j=1}^{n} \|y_j^*\|^q + \frac{1}{q} \sum_{j=n+1}^{m} \|y_j^*\|^q + \frac{1}{q'} \sup_{x^*} \sum_{j=1}^{n} |x^*(x_j)|^{q'}$

$+ \frac{1}{q'} \sup_{x^*}$

$$= \left[\frac{1}{q} \sum_{j=1}^{n} \|y_j^*\|^q + \frac{1}{q'} \sup_{x^*} \sum_{j=1}^{n} |x^*(x_j)|^{q'} \right] + \left[\qquad \right]$$

$$= \left(\sum_1^n \|y_j^*\|^q \right)^{1/q} w_{q'}(x_1 \dots x_n) + \left(\sum \|y_j^*\|^q \right)^{1/q} w_{q'}(x_{n+1} \dots x_m)$$

$\uparrow$

if we have picked s,t at the beginning so that these equalities hold ∇

Taking inf over best possible representations yields the claim. $\boxed{\equiv}$

<u>Lemma</u> $\pi_p(T) = \upsilon^*_{p'}(T)$ (Apply definition)

— The π_2 — exception —

<u>Prop</u> Let E be a finite dimensional Banach space. Then $\upsilon_2(S\, E \to X) \le \pi_2(S\, E \to X)$

(HW p-integral, show $\dim E < \infty \Rightarrow \upsilon_p \le I_p$)