

$$= \left[\frac{1}{q} \sum_{j=1}^n \|y_j\|^q + \frac{1}{q'} \sup_{x \sim} \sum_{j=1}^n |x^*(y_j)|^{q'} \right] + \left[\right]$$

$$= \left(\sum_{j=1}^n \|y_j\|^q \right)^{1/q} \omega_{q'}(x, \{y_j\}) + \left(\sum_{j=1}^n \|y_j\|^q \right)^{1/q} \omega_{q'}(x_{ur}, \{y_j\})$$

if we have picked s, t at the beginning so that these equalities hold ∇

Taking up over best possible representations yields the claim. \square

Lemma $\Pi_p(T) = \mathcal{U}_p^*(T)$ (Apply definition)

— The Π_2 -exception —

Prop let E be a finite dimensional Banach space
 Then $\mathcal{U}_2(S: E \rightarrow X) \leq \Pi_2(S: E \rightarrow X)$

(HW p -integral, show $\dim E < \infty \Rightarrow \mathcal{U}_p \leq I_p$)

Lemma let E be finite dimensional and $\varepsilon > 0$

Then there exist $m \in \mathbb{N}$ and

$\tilde{E} \subseteq \ell_2^m$ and $u: E \rightarrow \tilde{E}$ such that

$$(1-\varepsilon)\|x\| \leq \|u(x)\| \leq (1+\varepsilon)\|x\| \quad \forall x \in E$$

(Notation: $\text{dist}(E, \tilde{E}) \leq \frac{1+\varepsilon}{1-\varepsilon}$)

Proof: A δ -net $C \subset B_{E^*}$ is a subset S

such that

$$\begin{matrix} s, t \in S \\ s \neq t \end{matrix} \implies \|s-t\|_{E^*} > \delta$$

Since B_{E^*} is compact every δ -net has to be finite

Now take a maximal (finite) δ -net. S

Then for all

$$\boxed{x^* \in B_{E^*} \quad \exists s \in S \quad \|x^* - s\|_{E^*} \leq \delta}$$

Define $u: E \rightarrow \ell_2(S)$ $u(x) = (s(x))_{s \in S}$

Clearly $\|u(x)\| \leq \|x\|$ for all $x \in E$.

Let $\|x\|_E = 1$. Then there exists $x^* \in B_{E^*}$ with $|x^*(x)| = 1$. Let $s \in S$ such that

$\|s - x^*\|_{E^*} < \delta$. We get

$$|S^*(x)| = |S(x) - x^*(x) + x^*(x)|$$

$$\geq |S(x) - x^*(x)| + 1$$

$$\geq 1 - \delta \|x\| = 1 - \delta$$

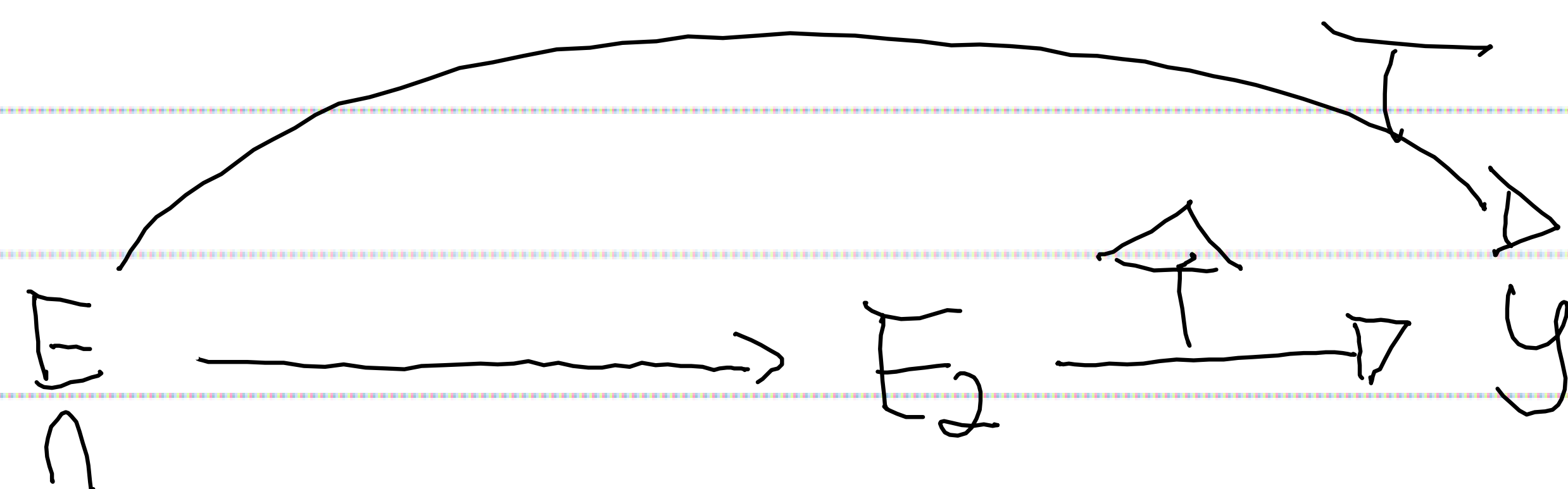
By homogeneity

$$(1 - \delta) \|x\| \leq \sup_{s \in S} |S(x)| \leq \|x\| \quad \forall x \in E \quad \square$$

Prop $\dim E < \infty \quad \pi_2(T: E \rightarrow Y) \leq \pi_2(T)$

~~Lemma~~ Let $E \subset F$ and $T: E \rightarrow Y$ be 2-summing. Then there exists $\tilde{T}: F \rightarrow Y$ which is also 2-summing.

Proof



We have a factorization

$$C(B_{E^*}) \longrightarrow L_2(B_{E^*}, \mu)$$

for T because

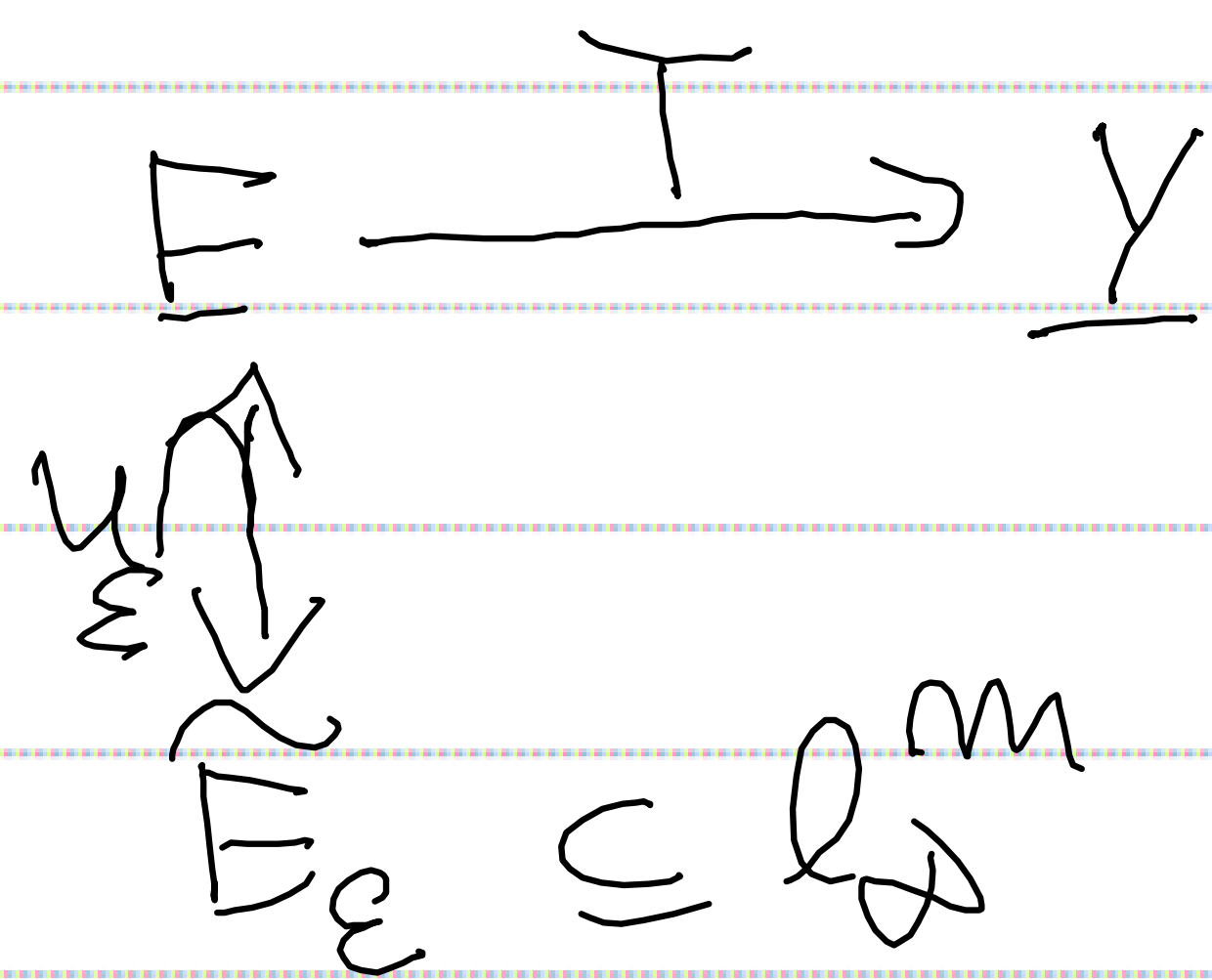
$$\|Tx\|_Y \leq \pi_2(T) \left(\int_{B_{E^*}} |x^*(x)|^2 d\mu \right)^{1/2}$$

and here

$$\|x\|_{E_2} = \left(\int_{B_E} |x^\nu(x)|^2 d\mu \right)^{1/2}$$

is the induced norm.

However, we want to change this



Then $T u_\varepsilon^{-1} : E_\varepsilon \rightarrow Y$ satisfies

$$\|T u_\varepsilon^{-1}\| \leq \|T\| \|u_\varepsilon^{-1}\|$$

\Rightarrow Grothendieck-Pietsch $\exists v_1, \dots, v_m$ on \mathbb{R}^m $\sum v_j^2 = 1$

$$\begin{aligned} \|T u_\varepsilon^{-1}(x)\|_Y &\leq \|T\| \|u_\varepsilon^{-1}\| \\ &\leq \|T\| \left(\sum_{j=1}^m v_j |\langle x, v_j \rangle|^2 \right)^{1/2} \\ &= \left(\sum_{j=1}^m |\langle x, \sqrt{v_j} v_j \rangle|^2 \right)^{1/2} \end{aligned}$$

Let $E_2 \subseteq \mathbb{R}^m$ be the subspace

given by $E_2 = \langle x, \sqrt{v_j} v_j \rangle : x \in \mathbb{R}^m$

Let $P : \mathbb{R}^m \rightarrow E_2$ be the orthogonal projection

Then

$$\| \underbrace{T u_\varepsilon^{-1} P_2}_{S} : \hat{E}_2 \rightarrow Y \| \leq \| u_\varepsilon^{-1} \| \tau_2(T)$$

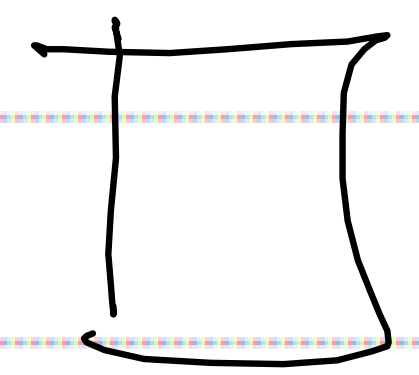
Maximum $\| x_j^* \| = s_j \sqrt{v_j}$ s.t. $s_j v_j = 1$

$$\left(\sum_{j=1}^m \| x_j^* \|_{E^*}^2 \right)^{1/2} \leq \left(\sum_{j=1}^m v_j \right)^{1/2} \sup_j \| s_j \|_{E^*} \leq \| u_\varepsilon \|,$$

Hence we get

$$\begin{array}{l} \cancel{E \rightarrow E^*} \\ \cancel{E \rightarrow E^*} \end{array} \left| \begin{array}{l} E \rightarrow E^* \xrightarrow{\cdot} Y \\ T = S W \\ = \sum_{j=1}^m x_j^* \otimes S(e_j) \in Y \end{array} \right.$$

$$\| U_2(T) \| \leq \| u_\varepsilon^{-1} \| \| u_\varepsilon \| \tau_2(T) \leq (1+\varepsilon) \frac{1}{1-\varepsilon} \tau_2(T)$$



Cor. $\dim E < \infty$

$$|\tau_1(ST)| \leq \tau_2(S) \tau_2(T)$$

Ex $T: X \rightarrow Y$ $S: Y \rightarrow X$ one of them finite rank

then

$$i) \|T(ST)\| \leq \pi_2(T) \pi_2(S)$$

$$ii) \pi_2(S) = \sup \{ \|T(ST)\| \mid \pi_2(T) \leq 1, T \text{ finite rank} \}$$

Proof: a) T finite rank $\begin{array}{ccc} X & \xrightarrow{S} & X \\ T(X) \subseteq E & \searrow S|_E & \end{array}$

$$\text{Then } \|T(ST)\| = \|T(S|_E T)\| = \|T(S|_E)\|$$

$$\leq \pi_2(T) \cup_2(S|_E)$$

$$\leq \pi_2(T) \pi_2(S|_E)$$

$$\leq \pi_2(T) \pi_2(S)$$

b) S finite rank: exercise

ii) i) shows \geq

\leq apply defunct bc $\pi_2(S) \leq \cup_2(S)$

□

Lemma A: $u: \ell_2^n \rightarrow \ell_2^n$

$$\pi_2(u) = \left(\sum_{j=1}^n |(e_j, u(e_j))|^2 \right)^{1/2}$$

Theorem (John) Let E be an n -dimensional Banach space

Then there exists a map $u: \mathbb{R}_2^n \rightarrow E$ such that

$$\|u\| \pi_2(u^t) = \sqrt{n}$$

($u(B_2^n) \subseteq E$ is the ellipsoid of maximal volume)

Proof By Lewis lemma there exist $u: \mathbb{R}_2^n \rightarrow E$ such that

$$\pi_2(u) \pi_2(u^t) = n$$

We may assume $\pi_2(u) = \sqrt{n}$ $\pi_2(u^t) = \sqrt{n}$.

By Pietsch factorization theorem we find

$$\begin{array}{ccccc} u & \mathbb{R}_2^n & \xrightarrow{v} & E_2 & \xrightarrow{w} & E & \xrightarrow{u^t} & \mathbb{R}_2^n \\ & \downarrow & & \uparrow & & & & \\ & C(u) & \xrightarrow{\quad} & L_2(\mu) & & & & \end{array}$$

such that $\|w\| = 1$ $\pi_2(w) = \sqrt{n}$. Wlog we may assume

$E_2 = \mathbb{R}_2^m$. Then we have

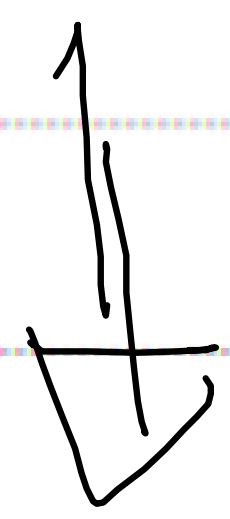
$$\pi_2(v: \mathbb{R}_2^n \rightarrow \mathbb{R}_2^m) = \sqrt{n} \quad \pi_2(u^t w: \mathbb{R}_2^n \rightarrow \mathbb{R}_2^m) = \sqrt{n}$$

and ~~$\pi_2(u^t w) = \sqrt{n}$~~

$$\underbrace{(u^t w)}_{a^t} v = u^t w v = u^t u = \text{id}$$

Key observation

$$n = \text{tr}(a^{-1}a) = \|a^{-1}\|_2 \|a\|_2 = \sqrt{n} \sqrt{n} \quad (o)$$



$a^{-1} = a^*$ hence a is unitary.

Indeed, we may use inner product $\ell_2^{n^2}$

$$(b, a) = \sum_y \overline{b_y} a_y$$

Then $(b, a) = \|b\|_2 \|a\|_2$ only holds

for $b = \lambda a$ $\lambda \in \mathbb{K}$.

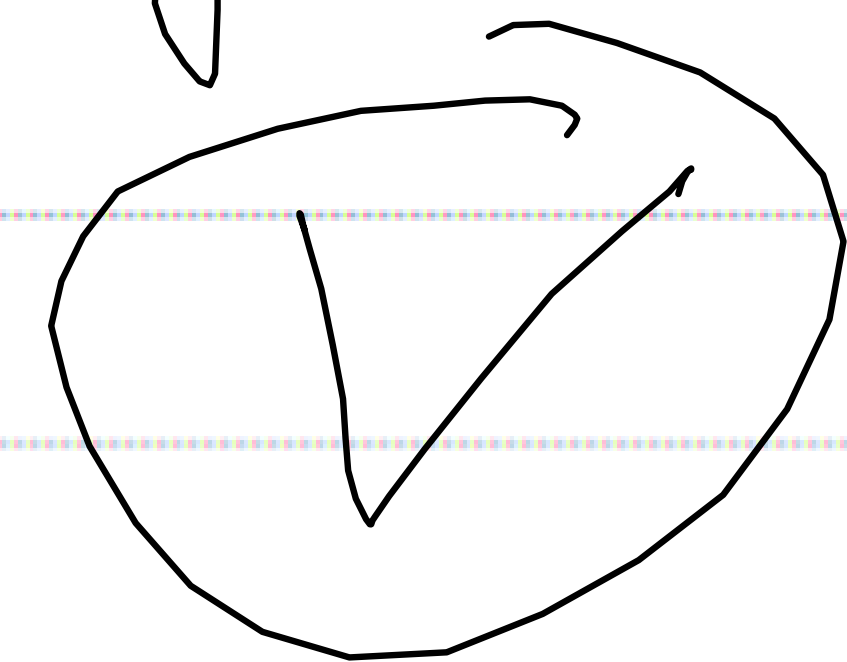
(⁴ Equality in Cauchy Schwarz)

$$\text{tr}(b^*a) = \sum_{jk} \overline{(b^*)_{jk}} a_{kj} = \sum_{jk} \overline{b_{kj}} a_{kj} = (b, a)$$

Thus (o) is satisfied for $a = v$

Therefore we see that $u = wv$ satisfies

$$\|u\| \leq 1 \quad \text{Tr}_2(u^*) \leq \text{Tr}$$



For the additional part we need another

Lemma B $\pi_2(u) \leq \sqrt{n} \|u\| \quad \forall u: \ell_2^n \rightarrow X \quad 1)$


all we need $\rightarrow \|u\|^\vee = \|\cup_1^\circ(v)\| \leq \sqrt{n} \pi_2(v) \quad \forall v: X \rightarrow \ell_2^n \quad 2)$
 Then we get $(\text{dual } \Leftrightarrow)$

~~$\pi_2(u) \leq \sqrt{n} \|u\|$~~

$\|u\| \leq 1 \quad \cup_1^\circ(u^\vee) \leq \sqrt{n} \pi_2(u^\vee) = n$

Then here

$\|u\| \cup_1^\circ(u^\vee) = n$ for $\alpha = \|u\|$

By uniqueness of ellipsoid of maximal volume we deduce the assertion 


Additional HW

Proof of Lemma B via Lemma A

1) $\pi_2(u) = \pi_2(u |_{\ell_2^n}) \leq \|u\| \pi_2(|_{\ell_2^n})$
 $= \|u\| \left(\sum_{i=1}^n \|e_i\|^2 \right)^{1/2} = \|u\| \sqrt{n}$

2) let $v: X \rightarrow \ell_2^n \quad u: \ell_2^n \rightarrow X$

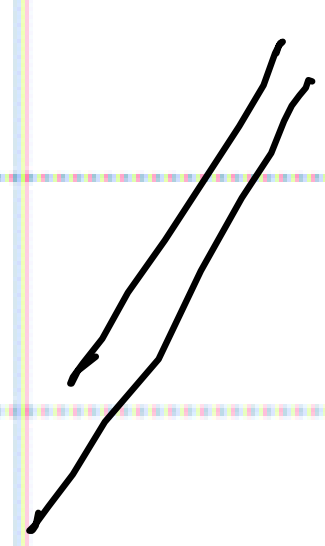
$|\pi_2(vu)| \leq \pi_2(v) \pi_2(u) \leq \pi_2(v) \sqrt{n} \|u\|$

Thus $\|u\|^\vee(v) \leq \sqrt{n} \pi_2(v)$. If in addition $\text{dual } \Leftrightarrow$
 then $\|u\|^\vee = \cup_1^\circ$ 

Proof of (bound A)

let $a: \ell_2^m \rightarrow \ell_2^n$ Then

$$\left(\sum_{i=1}^m \|a(e_i)\|_{\ell_2}^2 \right)^{1/2} \leq \pi_2(a) \sup_{\|x\|_2 \leq 1} \left(\sum_{i=1}^m |x^i(e_i)|^2 \right)^{1/2}$$



$$\leq \pi_2(a) \sup_{\|x\|_2 \leq 1} \left(\sum |x_i|^2 \right)^{1/2}$$

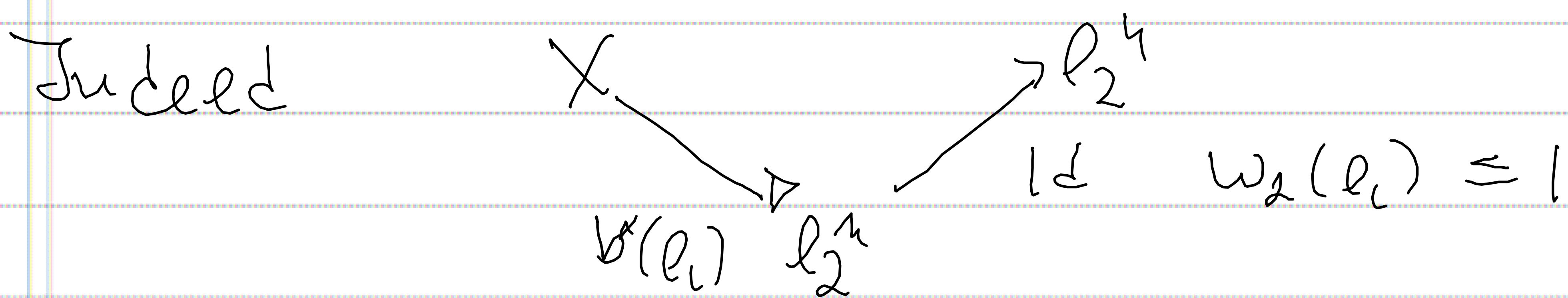
$$\boxed{\|a\|_{HS}}$$

$$\leq \pi_2(a)$$

Thus $\|a\|_{HS} \leq \pi_2(a)$

Observation $a: \ell_2^m \rightarrow X \quad \left(\sum_{i=1}^m \|a(e_i)\|_X^2 \right)^{1/2} \leq \pi_2(a)$

Observation $b: X \rightarrow \ell_2^m \quad \pi_2(b) \leq \mathcal{V}_2^0(b) \leq \left(\sum_{i=1}^m \|b(e_i)\|_{\ell_2}^2 \right)^{1/2}$



Hence we get

$$\|a\|_{HS} \leq \pi_2(a) \leq \left(\sum_{i=1}^m \|a^*(e_i)\|_{\ell_2}^2 \right)^{1/2} = \|a^*\|_{HS}$$

but $\|a\|_{HS} = \|a^*\|_{HS}$ and we are done \square

lemma let X, Y be a real n -dimensional Banach space.

and $T: X \rightarrow Y$ Then there are $x_1^x \dots x_m^x, y_1 \dots y_m$

such that

$$V_1^0(T) = \sum_{i=1}^m \|x_i^x\|_X \|y_i\|_Y, \quad T(x) = \sum_{i=1}^m x_i^x(x) y_i$$

and $m \leq$

Proof we consider the finite dimensional space

$(V_1^0(X, Y), V_1^0)$. Then

$$V_1^0(T) = \inf \sum_{k=1}^m \|x_k^x\| \|y_k\| \quad : \quad \{x_k^x \in X, y_k \in Y\}$$

$$= \sum_{k=1}^m \lambda_k \|x_k^x\| \|y_k\| \quad \left. \begin{array}{l} \lambda_k \leq 1 \\ \|x_k^x\| \leq 1, \|y_k\| \leq 1 \end{array} \right\}$$

Hence

$$\overline{\text{Ext}(B_{V_1^0})} \subseteq \{x^x \otimes y : \|x^x\| = 1, \|y\| = 1\}$$

these \implies are extremal points
(finite dimension, forget closure) \checkmark

Thus $B_{V_1^0}$ is the convex hull of its extreme points $\text{Ext} \subseteq \text{Rank}^{\text{all}} \text{ tensors}$ Dimension $V_1^0(X, Y)$

is n^2 Hence $n^2 + 1$ many points suffice.

Lemma Let $u: X \rightarrow Y$ den $X = u$ such that

$$\|u\| = 1 \quad \langle u, u \rangle = 1 \quad \|u\| = 1$$

Then there $x_1^* \dots x_m^* \quad m \leq n^2 \quad u = \sum x_j^* \otimes y_j$
 $y_1^* \dots y_m^* \quad m \leq n^2$

and $\|w(y_j)\|_X = \|y_j\|_Y$

$$\|w^*(x_j^*)\|_{Y^*} = \|x_j^*\|_{X^*}$$

Proof $L = \{u \in X \rightarrow Y : \langle u, u \rangle = 1\}$ is a $n^2 - 1$ dimensional affine hyperplane.

Claim $u \in \text{conv}(L \cap \{x^* \otimes y : \|x^*\| = 1 = \|y\|\})$

Indeed $u = \sum_{j=1}^m \alpha_j x_j^* \otimes y_j$ $\|x_j^*\| = 1 = \|y_j\|$
 $\sum \alpha_j \|x_j^*\| \|y_j\| = 1$

Then

$$1 = \langle u, u \rangle = \sum_j \langle x_j^* \otimes w(y_j) \rangle \leq \sum \|x_j^*\| \|w(y_j)\|$$

$$\leq \sum \|x_j^*\| \|y_j\| \leq 1$$

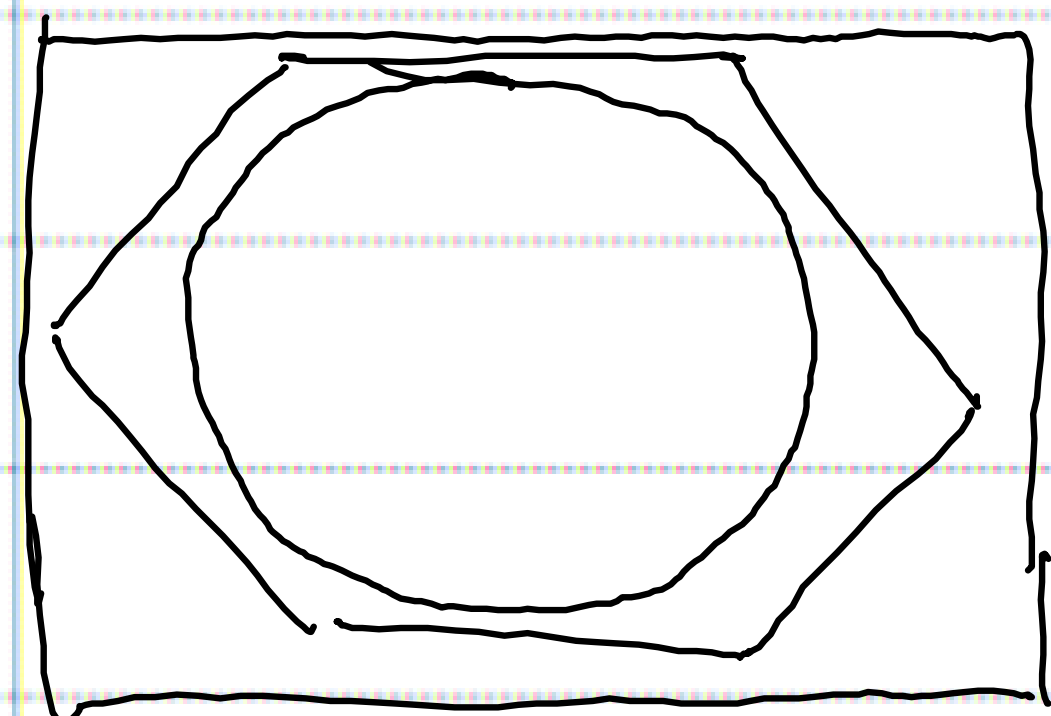
Thus we have equality and therefore $\|w(y_j)\| = \|y_j\|$
 $\|w^*(x_j^*)\| = \|x_j^*\|$ and

$$\langle x_j^*, w(y_j) \rangle = \|x_j^*\| \|w(y_j)\| = \|x_j^*\| \|y_j\| \quad \forall j$$

This means $\frac{x_j^* \otimes y_j}{\|x_j^*\| \|y_j\|} \in L$ □

→ Tomczak-Jaegermann $p=3/8$ using self adjoint element
 - (In Hilbert space this can be improved further!)

Comments



Hexagon

We need more than two

vectors to write $I = \sum x_j^* \otimes x_j$

I

II John's map $u = \text{Id}(\mathbb{Q}_2^n(1,1)) \rightarrow (\mathbb{R}^n, \|\cdot\|_E)$

$$u^{-1} = \sum_{j=1}^m d_j x_j^* \otimes x_j$$

$$|x_j^*| = \|x_j^*\| = 1$$

$$\|x_j\| = \|x_j\| = 1$$

Indeed use $w = u$ above and $\|u\| = \|u^{-1}\|$ then we get this

Sauer-Shelah lemma let $\dim E = n$. Then for

some subspace $F \subset E$ with $\dim F = \delta n$

$$\begin{array}{c} \mathbb{R}^n \xrightarrow{u} F \xrightarrow{u^T} \mathbb{R}^n \\ \cup \\ F \xrightarrow{\quad} F \xrightarrow{\quad} \mathbb{R}^n \end{array}$$

one can find a factorization with n -points !

(helpful tool).

Another interesting ellipsoid

$K = \mathbb{R}$ let γ_n be the measure

$$d\gamma_n(x) = e^{-\frac{\|x\|^2}{2}} \frac{dx}{\sqrt{2\pi}^n}$$

Then the coordinate functions

$$g_k(x) = x_k$$

are independent normalized gaussian variables

$$(N(0,1) \quad \mathbb{E}|g_k|^2 = 1 \quad \mathbb{E}g_k = 0)$$

lemma The norm

$$l(u) = \left(\int_{\mathbb{R}^n} \left\| \sum_{k=1}^n g_k(u) e_k \right\|_{\mathbb{X}}^2 d\mu \right)^{1/2}$$

is a right ideal norm for every real Banach space

\mathbb{X} .

Proof We have to show that $l(u\sigma) \leq l(u)$

for all contractions $\sigma: \ell_2^n \rightarrow \ell_2^n$.

let us first assume that σ is a unitary

then

$$\begin{aligned}
\ell(u_0) &= \left(\int_{\mathbb{R}^n} \left\| \sum_{k=1}^m g_k u_k(x) \right\|_{\Sigma}^2 d\mu(x) \right)^{1/2} \\
&= \left(\int_{\mathbb{R}^n} \|u_0(x)\|_{\Sigma}^2 d\mu(x) \right)^{1/2} \\
&= \left(\int_{\mathbb{R}^n} \|u_0(x)\|_{\Sigma}^2 e^{-\frac{|x|^2}{2}} \frac{dx}{\sqrt{(2\pi)^n}} \right)^{1/2} \\
&= \left(\int_{\mathbb{R}^n} \|y_0(y)\|_{\Sigma}^2 e^{-\frac{|y|^2}{2}} \frac{dy}{\sqrt{(2\pi)^n}} \right)^{1/2} \quad \begin{array}{l} y = \sigma x \\ |y| = |x| \end{array} \\
&= \ell(u).
\end{aligned}$$

Now let w be an arbitrary matrix. We can find σ such that

$$w\sigma = d \quad d \succcurlyeq 0 \quad \text{selfadjoint}$$

Let us diagonalize $d = v \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} v^{-1}$ v orthogonal

Then $\|d\| \leq 1$ means $\sup_j |\lambda_j| \leq 1$

The extreme points of $\mathcal{B}_{\mathbb{R}^n}$ are the vectors

(ϵ_j) $|\epsilon_j| = 1$ Thus

$$w = d\sigma = v \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} v^{-1} \sigma^{-1}$$

$$= \int v \begin{pmatrix} \epsilon_1 & & \\ & \ddots & \\ & & \epsilon_n \end{pmatrix} v^{-1} \sigma^{-1} d\mu(\epsilon)$$

$$\int |d\mu| = 1$$

Thus

$$\rho(u, w) = \int \rho(u, v \underbrace{D_\varepsilon v^{-1} \sigma}_{\text{unitary}}) d\mu(\varepsilon)$$

$$= \rho(u), \quad \square$$

Complex version: $\mathbb{H} = \mathbb{C}^n = \mathbb{R}^{2n}$

$$g_k = \frac{g_{k,2k} + i g_{k,2k+1}}{\sqrt{2}} \quad \text{complex gaussian.}$$

$$\rho(u) = \left(\mathbb{E} \left\| \sum_{k=1}^n g_k u(k) \right\|_{\mathbb{R}}^2 \right)^{1/2}$$

$$= \left(\int_{\mathbb{R}^{2n}} \left\| \frac{u(x+iy)}{\sqrt{2}} \right\|_{\mathbb{R}}^2 e^{-\frac{\|x\|^2 + \|y\|^2}{2}} \frac{dx dy}{(2\pi)^{2n}} \right)^{1/2}$$

Same argument applies, and now for complex unitaries.

Theorem The ellipsoid given by the ρ -norm is unique.

Pisier's theorem $v: \mathbb{R}^n \rightarrow \mathbb{R}^n$ $\left[\rho(v) \leq c(\|v\|) \rho(v) \right]$