

1 Finite dimensional spaces

Recall: all finite dimensional Banach spaces with fixed dimension "coincide" or all norms on \mathbb{R}^n are equivalent.

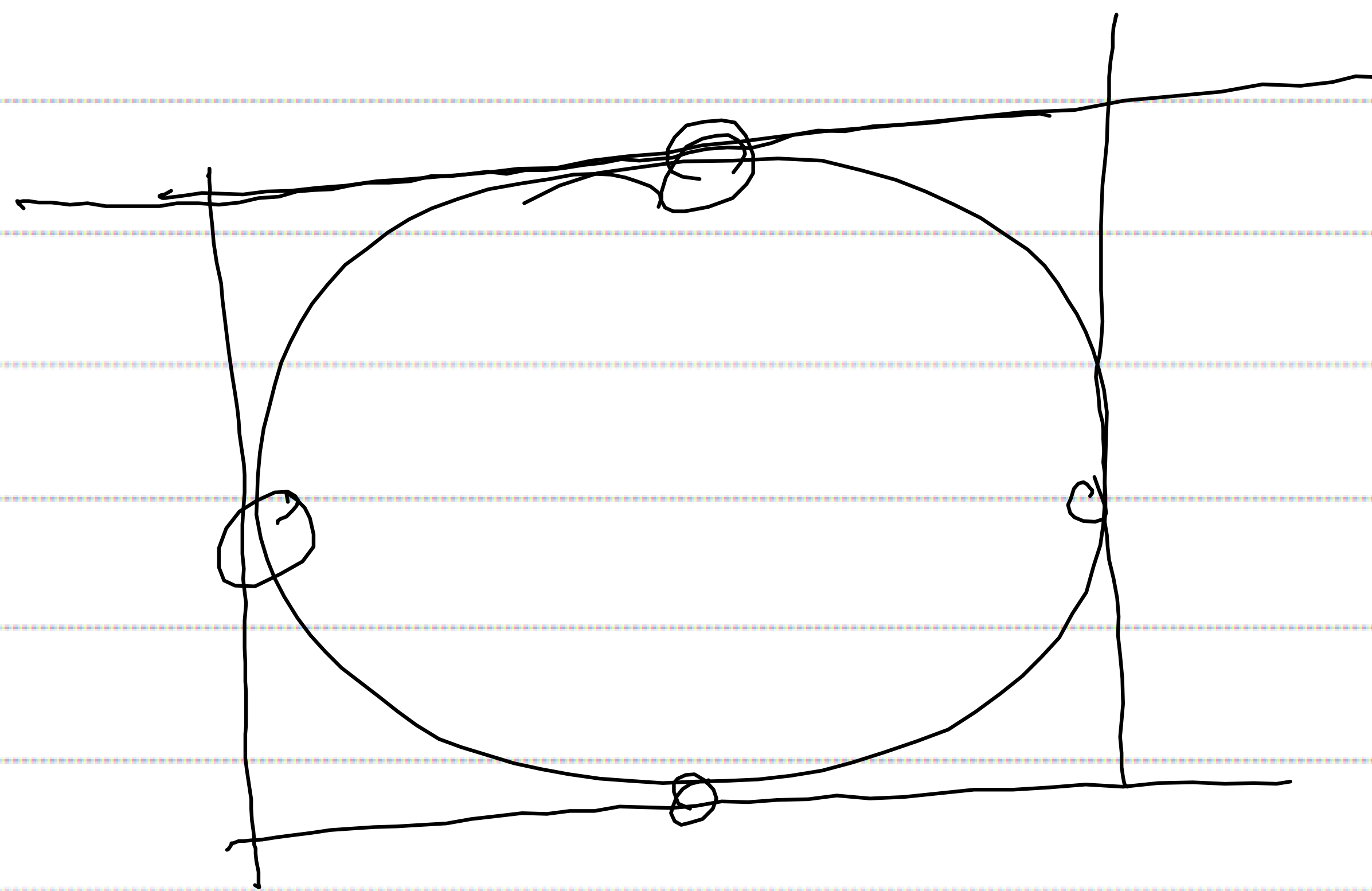
We want to prove something more precise

Theorem Let E be a n -dimensional Banach space. Then there exists vectors

$x_1, \dots, x_n \in E, x_1^*, \dots, x_n^* \in E^*$ such that

$$x_j^*(x_k) = \delta_{jk} \quad \|x_j^*\| = 1 = \|x_k\|$$

Geometrically



$((x_j), (x_j^*))$ is called Auerbach basis.

Determinants

A is a $n \times n$ matrix

$$\det A = \sum_{\sigma \in \text{Perm}_n} \text{sgn}(\sigma) \prod_{j=1}^n a_{j\sigma(j)} \quad (*)$$

has the following properties

$$\det AB = \det A \det B$$

$\det A = 0 \iff A$ has not full rank

$$\det A = \prod_{j=1}^n \lambda_j(A) \quad \text{if } A \text{ has } \lambda_j \text{ eigenvalues}$$

$$\det I = 1$$

$$\left\{ \begin{array}{l} p(t) = \det(I + tA) \quad \text{satisfies} \\ p'(0) = \text{tr}(A) = \sum_{k=1}^n a_{kk} \end{array} \right.$$

follows easily

(Other properties not so easy to obtain from $*$)

Def let α be a norm on $M_n = n \times n$ matrices

$$\alpha^*(A) = \sup \left\{ \frac{|\text{tr}(AB)|}{\|B\|} \mid \|B\| \leq 1 \right\}$$

$$\sum_{k,l} a_{kl} b_{kl} \quad (\text{upto } (i,i) \rightarrow (i,i) \text{ standard})$$

Lemma (Leuzs) let α be norm on \mathbb{K}^n . Then

there exist $a_0 \in \mathbb{K}^n$ invertible such that

$$\alpha(a_0) = 1 \quad \alpha^*(a_0^{-1}) = n$$

Proof: a_0 such that

$$|\det(a_0)| = \sup \{ |\det(a)| : \alpha(a) \leq 1 \}$$

Then $|\det(a_0)| \geq \frac{|\det(1)|}{\alpha(1)^n} > 0$ hence invertible.

$$|\det(a_0 + tb)| \leq \alpha(a_0 + tb)^n |\det(a_0)|$$

$$\Rightarrow |\det(1 + t a_0^{-1} b)| \leq \alpha(1 + t a_0^{-1} b)^n$$

$$\approx 1 + t \operatorname{tr}(a_0^{-1} b) + o(t^2)$$

$$|1 + t \operatorname{tr}(a_0^{-1} b) + t^2| \quad \boxed{t \in \mathbb{K}}$$

We may choose $t = \frac{1}{\sqrt{n}} \operatorname{sgn}(t)$ such that $\operatorname{sgn}(t) \operatorname{tr}(a_0^{-1} b) \geq 0$

$$\text{Hence } 1 + t |\operatorname{tr}(a_0^{-1} b)| + o(t^2) \leq 1 + \sqrt{n} \alpha(\operatorname{sgn}(t)b) + o(t^2)$$

$$\text{Hence } |\operatorname{tr}(a_0^{-1} b)| \leq n \alpha(b) + o(\sqrt{n}) \quad \leftarrow \begin{matrix} \mathbb{K} \rightarrow 0 \\ \leftarrow \end{matrix}$$

$$|\operatorname{tr}(a_0^{-1} b)| \leq n \alpha(b) \quad \text{Hence } \boxed{\alpha^*(a_0^{-1}) \leq n.} \quad \square$$

Application 1: There exists an n -Auerbach basis

Proof $\alpha(a) = \sup_{i=1, \dots, n} \|a(e_i)\|_{\mathbb{E}} \quad a: \mathbb{R}^n \rightarrow \mathbb{E} = (\mathbb{R}^n, \|\cdot\|)$

We find a invertible such that

$$a(e_i) = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix} x_i$$

$$x_i = a(e_i) \quad \text{satisfy} \quad \|x_i\| = 1$$

and $\alpha^*(a^{-1}) = n$

What is dual norm

$$\begin{aligned} \sup_{\alpha(a) \leq 1} \left| \sum_{i=1}^n b_i a_{ik} \right| &= \sup_{\|x_i\| \leq 1} \left| \sum_{i=1}^n \langle y_i, x_i \rangle \right| \\ &= \sum_{i=1}^n \|y_i\|_{\mathbb{E}^*} \end{aligned}$$

where $y_i \in \mathbb{E}^*$ is given by

$$(b_{i1}, \dots, b_{in})$$

Hence we find $y_i \in \mathbb{E}^*$

$$n = \sum_{i=1}^n \|y_i\|_{\mathbb{E}^*} = \sum_{i=1}^n \langle y_i, x_i \rangle = \text{tr}(\text{Id})$$

Hence $\langle y_i, x_i \rangle = \|y_i\|_{\mathbb{E}^*}$ for all $i=1, \dots, n$
and $\|x_i\| = 1$ for all $i=1, \dots, n$.

Moreover $\langle y_j, x_i \rangle = \delta_{ij}$ because we are working with orthonormal. Therefore

$$\|y_i\|_{\mathbb{C}^n} \geq \frac{\langle y_i, x_i \rangle}{\|x_i\|} \geq 1$$

If $\|y_i\|_{\mathbb{C}^n} > 1$ then $\sum \|y_i\|_{\mathbb{C}^n}^2 > n$ ∇ .

Thus $\|y_i\|_{\mathbb{C}^n} = 1$ for all $i=1, \dots, n$ \square

Real complex case works analogously for Lebesgue lemma and Aubach basis.

Definition A norm α on M_n has enough symmetries

\Leftrightarrow if there exists a compact group of isometries such that

$$i) \quad \alpha(g^{-1}ag) = \alpha(a)$$

$$ii) \quad \int_G g^{-1}ag \, d\mu(g) = \frac{\text{tr}(a)}{n} I \quad \leftarrow \text{Identity.}$$

Lemma $\alpha(I) \alpha^*(I) = n$

Proof let $b \in M_n$ such that $\boxed{\text{tr}(b) = \alpha^*(I)}$

$$\alpha(b) = I$$

(exists by Hahn Banach) Then

$$\alpha(g^{-1}bg) \leq 1, \quad \int \alpha(g^{-1}bg) \leq 1$$

$$\alpha\left(\int_G g^{-1}bg \, d\mu(g)\right) \leq 1$$

By property ii) we know that

$$\int g^r b g = \tau_n(b) \text{Id}$$

$$\text{and } \tau \left(\int g^r b g \right) \stackrel{\text{ii)}}{=} \tau(\tau_n(b) \text{Id}) = \tau_n(b) \tau(\text{Id}) =$$

$$\int \tau(g^r b g) = \tau(b) = \alpha^*(\text{Id})$$

Therefore $\lambda = \tau_n(b)$ satisfies

$$\alpha(\lambda \text{Id}) \leq 1$$

$$\stackrel{\text{ii)}}{\text{Id}} \alpha(\text{Id})$$

$$\stackrel{\text{ii)}}{\frac{\tau(n)}{n}} \alpha(\text{Id}) = \frac{\alpha^*(\text{Id}) \alpha(\text{Id})}{n} \gg \frac{\tau(\text{Id})}{n} \gg 1$$

We have equality, and the claim follows. \square

Two reasons for existence of $\alpha(a) \alpha^*(a^{-1}) = n$

Question? Uniqueness?

lemma \exists a non \emptyset such that

$$\alpha(AU) = \alpha(A) \quad \text{for all orthogonal unitary matrices.}$$

Then $\alpha(A) \alpha^*(A^{-1}) = n$ and $\alpha(A) = 1$
 $\alpha(B) \alpha^*(B^{-1}) = n$ and $\alpha(B) = 1$

implies that $B = AU$ for some U .

Proof: Wlog we may assume that A satisfies

$$\sup_{\alpha(\hat{A}) \leq 1} |\det \hat{A}| = |\det A|.$$

Consider $C = B^{-1}A$ (invertible)

Using Gram-Schmidt we may assume

$$C = \nabla U \quad U \text{ orthogonal}$$

$$\nabla \quad \text{upper diagonal}$$

Note that $|\det C| = |\det B^{-1}A| = |\det B|^{-1} |\det A| \geq 1$

$$\prod_{k=1}^n |\nabla_{kk}| \leq \left(\frac{\sum_{k=1}^n |\nabla_{kk}|}{n} \right)^n = \left(\frac{\text{tr}(\frac{D_\epsilon \nabla}{n})}{n} \right)^n$$

some diagonal with $|e_i| = 1$

$$\begin{aligned} & \text{geom / arithmetic} \\ & = \left(\frac{\text{tr}(D_\epsilon C U^{-1})}{n} \right)^n \\ & = \left(\frac{\text{tr}(C U^{-1} D_\epsilon)}{n} \right)^n \end{aligned}$$

However

$$\begin{aligned} |\operatorname{tr}(\bar{b}^T a \tilde{u}^T D_\epsilon)| &\leq \alpha^*(\bar{b}^T) \alpha(a \tilde{u}^T D_\epsilon) \\ &\leq n \alpha(a) \leq n \end{aligned}$$

Hence we have

$$|\operatorname{tr}(\bar{b}^T a \tilde{u}^T D_\epsilon)| \leq \alpha^*(\bar{b}^T) \alpha(a) \leq n$$

$\mathbb{K} = \mathbb{C}$ Instead of Gram-Schmidt

we use polar decomposition

$$\bar{b}^T a = w \left(\prod \lambda_j(|b^T a|) \right) u$$

and find

$$|\det(\bar{b}^T a)| = \left(\prod \lambda_j(|b^T a|) \right)^{1/n} > 1$$

$$\prod_{j=1}^n \lambda_j(|b^T a|) = \frac{|\operatorname{tr}(\bar{b}^T a u w)|}{n} \leq 1$$

Equality here implies $\lambda_j(|b^T a|) = 1$ hence

$\bar{b}^T a$ is already unitary.

~~$\mathbb{K} = \mathbb{R}$ $\bar{b}^T a = \nabla u$ ∇ is also complex matrix~~

~~and we still have $1 \leq |\det \nabla|^{1/n} \leq \frac{|\operatorname{tr}(\nabla)|}{n} \leq 1$~~

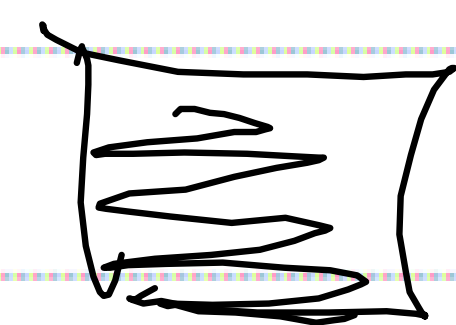
We shall now show that $\sqrt{(b^T a)^*(b^T a)}$ is real and

$$\|b^T a\| \text{ is real and } \| |b^T a| \| = \| b^T a \|$$

Hence $b^T a = |b^T a| \cdot \theta$ orthogonal

Hence argument from above shows that

$|b^T a|$ is unitary and hence $b^T a$ is orthogonal



Interpretation: The ellipsoid

$$E = a(B_2^n) \quad B_2^n = \{x \in \mathbb{R}^n, \|x\|_2 \leq 1\}$$

is independent of the choice of a with

$$\alpha(a) = 1 \quad \alpha^*(a^T) = n$$

Application: The ellipsoid $E \subset B_{rE} = \{x : \|x\|_E \leq r\}$
of maximal volume satisfies

$$\alpha^*(\bar{u}^T) = n \quad \text{where } \alpha = \|\cdot\|$$

Question What is α^* ?

Same remark holds for ellipsoid of minimal volume containing B_{rE} . Why?

Some examples

$$\ell_p^m = (\mathbb{K}^m, \|\cdot\|_p)$$

$$\|x\|_p = \left(\sum_{k=1}^m |x_k|^p \right)^{1/p}$$

$$\alpha(u) = \|u: \ell_2^m \rightarrow \ell_p^m\|$$

Note α is a right ideal norm

$$\alpha(uw) \leq \alpha(u) \|w\|_{\ell_2^m \rightarrow \ell_2^m},$$

$$G = \lambda^{-1,1} S^m \times \text{Perm}_m \subseteq U(\ell_2^m) = \text{unitary group}$$

$$\varepsilon \mapsto D_\varepsilon = \begin{pmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_m \end{pmatrix}$$

$$\pi \mapsto M_\pi(\varepsilon_i) = \varepsilon_{\pi(i)} \quad \text{permutation matrix}$$

Note $D_\varepsilon M_\pi D_\varepsilon^{-1}(\rho_i) = \varepsilon_{\pi(i)} \rho_{\pi(i)}$

$$= D_{\varepsilon_\pi} M_\pi$$

$$\varepsilon_\pi(\rho_j) = \varepsilon_j \rho_{\pi^{-1}(j)}$$

Hence $\{ D_\varepsilon M_\pi : \varepsilon \in \lambda^{-1,1}, \pi \in \text{Perm}_m \}$ is indeed

○ a subgroup, and Haar measure = counting meas

We need

$$\alpha(g^T u g) = \alpha(u)$$

Indeed $M_{\frac{1}{g}}$ and D_g are isometries on ℓ_p^m

and hence $\alpha(g^T u g) = \alpha(u)$ (using left multiplication.)

Then The ellipsoid of maximal volume in ℓ_p^m

is given by

$$B_2^m = B_{\ell_2^m} \subseteq B_{\ell_p^m} \quad \text{for } p > 2$$

$$n^{\frac{1}{2}-\frac{1}{p}} B_2^m \subseteq B_{\ell_p^m} \quad \text{for } 1 \leq p \leq 2$$

Proof We know that $\alpha(\text{Id}) \alpha^*(\text{Id}) = n$

Thus it suffices to calculate

$$\alpha(\text{Id} : \ell_2^m \rightarrow \ell_p^m) = \|\text{Id} : \ell_2^m \rightarrow \ell_p^m\|$$

For $p > 2$ we have $(\sum |x_i|^p)^{1/p} \leq (\sum |x_i|^2)^{1/2}$

and this is attained for $x = (1, 0, \dots, 0)$

For $1 \leq p \leq 2$ we have $(\sum |x_i|^p)^{1/p} \leq n^{\frac{1}{p}-\frac{1}{2}} (\sum |x_i|^2)^{1/2}$
and this is attained for $x = (1, \dots, 1)$ \square

Question? what is optimal u ? (in following sense)

$$d(\ell_2^m, \ell_p^m) = \text{dist}(\ell_2^m, \ell_p^m) = \inf \{ \|u\| \|u^{-1}\| : u \text{ isomorphism} \}$$

The examples above suggest to simply take

$u = \text{Id}$ and then we get

$$\boxed{d(\ell_2^m, \ell_p^m) \leq n^{|\frac{1}{2} - \frac{1}{p}|}}$$

How can we show this is sharp \rightarrow Type, Cotype

Definition An ideal of norms is an assignment

$$\alpha_0(E, F) \longrightarrow \alpha(E, F) \subseteq \mathcal{L}(E, F)$$

such that:

- $\alpha(E, F)$ is a set for fixed E, F

- $\alpha(E, F)$ carries a norm α

- $\alpha(RST) \leq \|R\| \alpha(S) \|T\|$

$$\text{and } R \in \mathcal{L}(F, G) \quad S \in \alpha(E, F) \quad T \in \mathcal{L}(D, E)$$

$$\text{implies } RST \in \alpha(E, F)$$

- For any rank one map

$$T(e) = e^*(e) f \quad \text{one has}$$
$$e^* f$$

$$\alpha(Te^*f) = \|e^*\| \|f\|,$$

Proposition Let α be an ideal

Let $1 \leq p, q \leq n$ Then the
for $\alpha(l_p^m, l_q^n)$
ellipsoid of maximal volume is given by

$$\frac{B_{l_2^{n^2}}}{\| \cdot \|_{l_2^{n^2}}} \rightarrow \alpha(l_p^m, l_q^n) \|$$

Proof: We have to identify a suitable group

$$G = G_1 \times G_1 \subseteq \alpha(l_p^m, l_q^n)$$

where $G_1 = \{ D_\varepsilon \Pi_\pi : \varepsilon \in \mathbb{R}_+, \Pi_\pi \in \text{Perms} \}$

$$\mathcal{P}(D_\varepsilon \Pi_\pi, D_\varepsilon \Pi_\pi)(R) = D_\varepsilon \Pi_\pi R D_\varepsilon \Pi_\pi$$

Obviously

$$B(T: l_2^{n^2} \rightarrow \alpha(l_p^m, l_q^n)) \text{ satisfies}$$

$$B(g^{-1} T g) = B(T)$$

It suffices to prove $\int_G g^{-1} T g \, dg = \tau(T) \text{Id}$.

Let $T: \mathbb{R}^{n^2} \rightarrow \mathcal{V}(\mathbb{R}^n, \mathbb{R}^n)$ ↙ has denom n^2

$$T(e_{ij}) = \sum_{k \neq l} t_{ijkl} e_{kl}$$

assume

$$\int g^{-1} T g \, dg = T$$

$$\text{Then } T(\varepsilon_i \varepsilon_j e_{ij}) = \varepsilon_i \varepsilon_j \sum_{k \neq l} t_{ijkl} e_{kl}$$

$$\parallel$$

$$g T(e_{ij}) = \sum_{k \neq l} t_{ijkl} \varepsilon_k \varepsilon_l e_{kl}$$

$$i \neq k \quad \varepsilon_i = \begin{cases} -1 & i \\ 1 & \text{all other} \end{cases} \quad \varepsilon_0 = 1$$

$$\text{Then we get } - \sum_{k \neq l} t_{ijkl} e_{kl} = \sum_{\substack{k \neq l \\ k \neq i}} t_{ijkl} e_{kl} - t_{ijie} e_{ie}$$

$$\sum_{k \neq i} 2 t_{ijkl} e_{kl} = 0$$

$$\Rightarrow t_{ijkl} = 0 \quad \text{if } i \neq k$$

Similarly

$$t_{ijkl} = 0 \quad \text{if } j \neq l$$

$$T(e_{ij}) = t_{ij} e_{ij}$$

$$T(\pi^*(e_{ij})) = T(e_{\pi(i), j}) = t_{\pi(i), j} e_{\pi(i), j}$$

$$\parallel \quad \pi^* T(e_{ij}) = t_{ij} e_{\pi(i), j}$$

Hence $t_{\pi(i),j} = t_{ij}$ for all i

Similarly $t_{i,\pi(j)} = t_{ij}$

This means $t_{ij} = t_{ji}$ is a constant

$$\boxed{T = t_{11} I_d}$$

□

Similar situation

For matrices define

$$\|a\|_p = \left(\sum_{j=1}^n |a_j|^p \right)^{1/p}$$

This space
is called

S_p^n

where $s_j(a) = d_j(|a|)$ $|a| = \sqrt{a^* a}$

are the singular numbers

Prop The ellipsoid of maximal volume in

S_p^n is

$$B_{p,2} \subseteq B_{S_p^n}$$

$p \geq 2$

$$n^{1/2 - 1/p} B_{p,2} \subseteq B_{S_p^n}$$

$1 \leq p \leq 2$

Proof HW.

□