It was Grothendieck who discovered that norm on tensor products of Banach spaces provide a powerful tool in studying these spaces, and moreover to tell different spaces apart. Grothendieck certainly used functorial properties such as injectivity and surjectivity of certain norms. After completing his fundamental work on tensor norms in Banach spaces he turned his attention to algebraic geometry (and never actively returned to Banach space theory).

Before we study norms on Banach spaces, let us first consider the algebraic tensor product of two finite dimensional vector spaces $X$ and $Y$. We may assume that $X$ has dimension $n$ and $Y$ has dimension $m$, and that $x_1,...,x_m$, and $y_1,...,y_m$ are basis. Then $X \otimes Y$ is simply a vector space of dimension $nm$. An elementary tensor

\[ x \otimes y \Rightarrow \sum_{i,j} x(i) y(j) \begin{array} {c} x_i \\ y_j \end{array} \]

is best understood as the matrix product of a row and a column matrix

\[ x \otimes y = \begin{pmatrix} x(1) \\ \vdots \\ x(n) \end{pmatrix} \begin{pmatrix} y(1) & \cdots & y(m) \end{pmatrix} \]

In other words $\mathbb{R}^n \otimes \mathbb{R}^m \cong L(\mathbb{R}^m, \mathbb{R}^n)$ as vector spaces.
Definition: Let $X, Y$ be Banach space. The algebraic tensor product $X \otimes Y$, is a subspace of $L(Y^*, X)$. For an elementary tensor $x \otimes y$, the corresponding linear map is given by

$$T_{x \otimes y}(y^*) = y^*(y)x.$$ 

Proposition: The closure of $T(X \otimes Y)$ in $L(Y^*, X)$ is exactly the closure of the set of finite rank, weak*-continuous maps.

Definition: $X \otimes Y$ is the completion of the algebraic tensor product with respect to the norm induced by $T$.

Proof (of Proposition): Let us assume that $T: Z \rightarrow X$ has finite rank. Then we may fix an Auerbach basis for the finite-dimensional space $T = T(Z)$. Let $(y_i^n)$, $(x_i)$ this basis. Then we see that $T_i^n = x_i^n \circ T$ is a continuous linear functional and

$$T(\tilde{z}) = \frac{1}{n} \sum_{i=1}^{n} T_i^n(\tilde{z}) x_i$$

is in the range of $\overline{\mathcal{T}}: \mathcal{Z} \otimes X \rightarrow L(Z, X)$

$$\overline{\mathcal{T}}(\mathcal{Z} \otimes X)(\tilde{z}) = \mathcal{Z}(\tilde{z}) X.$$

We note that for a finite rank map $T: Z \rightarrow X$ the double adjoint still satisfies $T(\mathcal{Z}) = \mathcal{Z}$. In our situation $\mathcal{Z} = Y^*$ and $T: Y^* \rightarrow X$ is finite rank and w*-continuous.
Remark: For $X=H$, $Y^*=H$, we find the closure of finite rank maps in $L(H) = B(H)$. The closure of finite rank maps are exactly the compact operators. Recall that $T$ from $X$ to $Y$ is compact if $T(B_X)$ is relatively compact in $Y$. For arbitrary Banach spaces the range of $X\hat{\otimes} X$ in $L(X)$ need not coincide with the space of compact operator, but this is hard to prove (Per Enflo ~72).

Remark: For $X=H$, $Y^*=H$, we find the closure of finite rank maps in $L(H) = B(H)$. The closure of finite rank maps are exactly the compact operators. Recall that $T$ from $X$ to $Y$ is compact if $T(B_X)$ is relatively compact in $Y$. For arbitrary Banach spaces the range of $X\hat{\otimes} X$ in $L(X)$ need not coincide with the space of compact operator, but this is hard to prove (Per Enflo ~72).

Remark: $X\hat{\otimes} Y$ can be embedded in $L(X,Y)$ and this embedding also yields . Similarly, we could have embedded $X \otimes Y$ in $L(X^*,Y)$ and still get the same result. In all cases the norm of a tensor $z = \sum x_i \otimes y_i$ is given by

$$
\|z\| = \sup \left\{ \sum_1^n |x_i(y_i^*)y_i(y_i)| : \|x_i\| \leq 1, \|y_i\| \leq 1 \right\}
$$

Remark: $\varepsilon$ is called smallest and/or injective tensor norm.
Definition: A tensor norm \( \alpha \) is an assignment of a norm on the algebraic tensor product \( X \otimes Y \) such that

i) \( \alpha(x \otimes y) = \|x\| \|y\| \) for all \( x, y \in X, Y \)

ii) \( \alpha(R \otimes I)(x \otimes y) \leq \|R\| \|I\| \alpha(x \otimes y) \)

We denote by \( X \hat{\otimes} Y \) the completion of the algebraic tensor product with respect to \( \alpha \).

Remark: Sometimes we consider left or right tensor norms where \( \alpha((R \otimes I)(x \otimes y)) \leq \|R\| \|I\| \alpha(x \otimes y) \) holds for all \( R : X_1 \rightarrow X_2 \) (but not all \( S : Y_1 \rightarrow Y_2 \)) or even only for special \( S \), ... 

Lemma: The injective tensor norm is the smallest tensor norm.

Proof: Let \( x^* : X \rightarrow \mathbb{K} \) be linear functionals and \( z = \sum_{i=1}^{n} x_i \otimes y_i \). Then

\[
|\sum_{i=1}^{n} x^*_i(x) y_i(y_i) | \leq \|x^*\| \|y^*\| \sum_{i=1}^{n} \|x_i\| \|y_i\|
\]

Taking sup implies assertion.

Definition-Proposition: The largest tensor norm is given by

\[
\|z\| = \sup \sum_{i,k} \frac{2}{r_i} \|x_i\| \|y_{i,k}\|
\]

\[ z = \sum_{i,k} x_i \otimes y_{i,k} \]
Proof: This is obviously the largest possible one. Moreover

$$\alpha(x \otimes y) \leq \|x\| \|y\|$$

Let \(x, y\) be norm-attaining functionals.

Then for every representation \(m\)

$$x \otimes y = \sum x_k \otimes y_k$$

we have

$$\|x\| \|y\| = \|x \otimes y\| (x \otimes y)$$

$$= \|x \otimes y\| \left( \sum_{k=1}^{n} x_k \otimes y_k \right)$$

$$\leq \|x\| \|y\| \| \sum_{k=1}^{n} x_k \otimes y_k \|$$

Thus taking the infimum still gives the lower bound

$$\|x\| \|y\| \leq \|x \otimes y\| \leq \|x\| \|y\|$$
In many situations Hilbert spaces are much easier to understand than arbitrary Banach spaces (exception the invariant subspace problem). In Banach space theory many ideas have been developed by comparing properties of Banach spaces by properties of Hilbert spaces. In particular, one may consider those linear maps $T$ from $X$ to $Y$ which factor through a Hilbert space in the form $T = vw$, where $w$ is a linear map from $X$ to $H$ and $v$ from $H$ to $Y$. The expression

$$
\gamma_2(T) = \sup \left\{ \| w \| \ : \ w : X \to H \right\} \sup \left\{ \| v \| \ : \ v : H \to Y \right\}
$$

happens to be a norm. The 'analogue' of this norm for tensors is given in the following definition:

**Definition:** For a tensor $z = \sum x \otimes y_i$ we define

$$
\gamma_2(z) = \inf \sup \left\{ \frac{m}{\| x \|} \sup_{1 \leq i \leq m} (v(x_i))^2 \sup_{1 \leq j \leq m} (y_j)^2 \right\} \| z = \sum x \otimes y_i
$$

**Lemma:** $\gamma_2$ is a norm.

**Proof:** We have to use

$$
ab = \max_{t > 0} \frac{(at)^2 + (bt)^2}{2}
$$

for two tensors $z_1$ and $z_2$ and $\epsilon > 0$. 

We can find rep's such that

\[ \gamma_2(z) \geq \frac{1}{2} \left[ \sup_x \sum_{k=1}^{n} \left| x^x (x_k) \right|^2 + \sup_y \sum_{k=1}^{m} \left| y^y (y_k) \right|^2 \right] \]

\[ \gamma_2(z^2) \geq \frac{1}{2} \left[ \sup_x \sum_{k=1}^{n} \left| x^x (x_k) \right|^2 + \sup_y \sum_{k=1}^{m} \left| y^y (y_k) \right|^2 \right] \]

\[ z' = \frac{1}{2} \sum_{k=1}^{n} x_k \otimes y_k \quad z = \frac{1}{2} \sum_{k=1}^{m} x_k \otimes y_k \]

Then \[ z' + z^2 = \sum_{k=1}^{n} x_k \otimes y_k + \sum_{k=1}^{m} x_k \otimes y_k \]

and hence \[ x_k = \{ x_{k-n} \} \quad y_k = \{ y_{k-m} \} \quad k \leq n \]

we get

\[ \gamma^2(z' + z^2) \leq \sup_x \left( \sum \left| x^x (x_k) \right|^2 + \sum \left| x^x (y_k) \right|^2 \right) \]

\[ \leq \frac{1}{2} \left[ \sup_x \sum_{k=1}^{n} \left| x^x (x_k) \right|^2 + \sup_y \sum_{k=1}^{m} \left| y^y (y_k) \right|^2 \right] \]

\[ \leq \frac{1}{2} \left[ \sup_x \sum_{k=1}^{n} \left| x^x (x_k) \right|^2 + \sup_y \sum_{k=1}^{m} \left| y^y (y_k) \right|^2 + \sup_y \sum_{k=1}^{m} \left| y^y (y_k) \right|^2 + \sup_y \sum_{k=1}^{m} \left| y^y (y_k) \right|^2 \right] \]

\[ \leq (H3) \gamma_2(z') + (H3) \gamma_2(z^2) \]

Sending \( \varepsilon \rightarrow 0 \) yields the assertion
Theorem (Grothendieck) There exists a constant $K > 1$ such that

$$\Pi(\ell^n) \leq K\, \chi_2(\ell^n)$$

for every tensor $z$ in $\ell^n \otimes \ell^m$.

Lemma: i) Let $x_1, \ldots, x_n$ Then

ii) Let $z$ be a tensor in $\ell^n \otimes \ell^m$. Then

$$\chi_2(z) = \| m_z \|_{m \to m} \| m_z(a) \|_{m \to m}$$

Proof:

Part i) Let $S : \ell^m_1 \to \ell^m_2$. Then

$$\| S \| = \sup_{k \in \ell^{1\times m}} \| S(k e) \|_{2} = \sup_{k \in \ell^{1\times m}} \left( \sum_{j=1}^{n} \left| \langle k, e_j \rangle \right|^2 \right)^{1/2}$$

$+$ $k \in \ell^n$ (unit vectors) are extreme points

Now we consider $z = \frac{1}{n} \sum_{j=1}^{n} e_j \otimes e_j$ and the corresponding linear map

$$S = T_{z} : \ell^m_1 \to \ell^m_2$$

$$T_{z} (k e) = \sum_{j=1}^{n} \langle k, e_j \rangle e_j = \sum_{j=1}^{n} \langle k, e_j \rangle e_j$$

Hence we find a 1-1 correspondence between $S$ and the matrix $(e_j S(k e))$

$-$ the matrix $(e_j S(k e))$

such as $\| S \| = \sup_{k \in \ell^{1\times m}} \left( \sum_{j=1}^{n} \left| \langle k, e_j \rangle \right|^2 \right)^{1/2}$
Part ii) is a little more challenging. First, given a matrix \((m_j)\) corresponding to

\[
Z = \bar{Z} m_j \otimes \bar{q}_j
\]

we noted that \(\bar{z}_2(\bar{q}_j) = \bar{z}_2 (\bar{q}_j : l^m_j \to \bar{Q}^m)\)

\[
= \sup_{m_{i,j}} \sup_{\|h_i\|_H} \sup_{\|\bar{y}_j\|_H} \|h_i, y_j\|
\]

where the sups are taken over all \((fd)\) Hilbert spaces.

Indeed \(T_2 = RS; S : l^m_1 \to H, R : H \to l^m_2\). Then as above

\[
\|R x\| = \|R z\| = \sup_{j} \|R (\bar{q}_j)\|
\]

\[
\|S x\| = \sup_{0} \|S (\bar{q}_j)\|
\]

Let us show that

\[
\|m_{2i}\| \leq \bar{z}_2(\bar{Q})
\]

Assume \(\bar{z}_i = (\bar{h}_i, \bar{y}_j)\), \(\|\bar{h}_i\|, \|\bar{y}_j\| \leq 1\)

In fact we may define \(u : l^m_2 \to l^m_2 (H)\) \(u(x) = (\alpha \otimes h_i)\)

and \(w : l^m_2 \to l^m_2 (H)\) \(w(x) = (\beta_2 \otimes \bar{q}_j)\)

Then

\[
\|w (\alpha \otimes h_i)\|_H \leq \|w\|_H \|\alpha\|_H \|h_i\|_H
\]
and \( (e_j, w(a \otimes b) w(e_i)) \)
\[ = a_y (h \otimes j) \]

For the converse, we assume that
\[ \| [m_y a_j] \|_{L_m} \leq \| a_y \|_{L_m} \]

Here we need another result.

**Theorem:** Let \( S : l_2^m \rightarrow l_2^m \) be a linear map. Then
\[ \| \gamma(ST) \| \leq \gamma_2 (\gamma) \]

If \( \exists x \in l_2^m \) \( \sum_i x_i^2 = 1 = \sum_j y_j^2 \) and \( a : l_2^m \rightarrow l_2^m \)

Such that
\[ S x_j = B_j a_y x_i \]

Let's first continue the proof. Let \( \gamma : l_2^m \rightarrow 0 \)

be such that \( \gamma(2) = \gamma_2 (\gamma) \). Then \( \gamma \) is implemented to be a matrix and we may assume
\[ \gamma(2) = \gamma(S^T 2) \]

By the theorem \( S = B_j a_j a_y \) and hence
\[ \gamma(S^T) = 2 \sum_j S_j z_j = \sum_i B_j a_j z_j \]
\[ = \langle b, m_2 (a^*) \rangle \leq \| m_2 (a^*) \| \| m_2 (b) \| \]
Thus \( r_2(z) \leq 1 \nabla z \nabla \)

**Proof (Theorem):** We assume

\[
\frac{1}{2} \, \nabla \left( \text{ST}_x \psi \right) \leq \sup_{i=1}^{n} \left( \frac{1}{2} \chi(x_i)^2 \right) \leq \sup_{i=1}^{n} \left( \frac{1}{2} \chi(x_i)^2 \right)
\]

\[
\leq \sup_{y} \frac{1}{2} \, 2 \chi(x)^2 + 1 \chi(y)^2
\]

In \( \mathbb{R}^m \), we define the cone of elements spanned by

\[
\sum_{y} = \frac{1}{2} \chi(x)^2 + 1 \chi(y)^2 - \text{tr} \left( S \times \psi \right)
\]

\[ C = \{ x \in \mathbb{R}^m \mid \text{tr} \geq 0 \}
\]

This cone is disjoint from

\[ C_\leq = \{ x : h_{x,m} \rightarrow \mathbb{R} | \sup_S \psi(y) < 0 \}
\]

By the Hahn-Banach theorem there exists a linear functional \( \psi \) such that

\[
\sup_{x} \psi(x) \leq t \leq \inf_{x} \psi(x)
\]

Since \( C_\leq \) is a cone \( t > 0 \), hence \( t = 0 \)
Now \( \sum_{y} \psi_y^2 \leq 0 \) when \( \sum_{y} \psi_y < 0 \),
implies \( \psi_y > 0 \) for all \( y \).
In addition,
we may assume \( \sum_{y} \psi_y = 1 \).
We obtain
\[
\frac{1}{2} \text{Tr}(S T_{x \psi_y}) \psi_y \leq \frac{1}{2} \sum_{y} \left[ \frac{1}{d} \left( \psi_y^2 \bar{d}^2 \right) + \frac{1}{2} \frac{1}{d} \left( \bar{d}^2 \psi_y^2 \right) \right]
\]
\[
= \frac{1}{2} \left[ \frac{1}{d} \sum_{y} \left( \psi_y^2 \bar{d}^2 \right) + \frac{1}{2} \frac{1}{d} \left( \bar{d}^2 \psi_y^2 \right) \right] = \frac{1}{2} \left( \sum_{y} \psi_y \right) \psi_y
\]
The we define
\[
\lambda_y = \sqrt{\frac{2}{d} \psi_y} \quad \beta_y = \sqrt{\frac{2}{d} \psi_y}
\]
Using \( ab = \sqrt{\frac{1}{2} \left( a^2 + \frac{1}{2} b^2 \right)} \) we ge
\[
|\text{Tr}(S T_{x \psi_y})| \leq \left( \sum_{y} \left( \frac{2}{d} \psi_y \right) \right)^{\frac{1}{2}} \left( \sum_{y} \left( \frac{1}{d} \psi_y \right) \right)^{\frac{1}{2}}
\]
Observe that
\[
\text{Tr}(S T_{x \psi_y}) = \langle x, S \psi_y \rangle
\]
Thus \( \sum_{y} \bar{d}^2 = \alpha \) is a matrix of norm \( \leq 1 \)
and \( S = D_2 a D_3 \) as claimed. Here \( \alpha = 0 \) for \( \xi = 0 \)
means that we don't need that entry.
Definition: A mean 0 gaussian random variable is given by
\[ g = \sum h_i g_i \quad h_i \in \mathbb{R} \]
where \( g_i \) are independent normalized gaussian variables.

Lemma: \( (g, g') = \sin(\frac{\pi}{2} E(\text{sgn}(g) \text{sgn}(g'))) \) where \( \text{sgn} \) is the signum function.

Proof: Recall that \( g_i \in \mathbb{R} \) are just the coordinate functions, and that the gaussian measure is given by \( \frac{1}{(2\pi)^{n/2}} e^{-\frac{x^2}{2}} dx \). The angle of two gaussian random variables
\[ g = \sum h_i g_i \quad g' = \sum h'_i g_i \]
is given by \( \sin \theta_0 = (g, g') = (h, h') \) with \( \theta_0 \in [-\pi, \pi] \). By rotation invariance we may assume that \( g = g_1 \), and \( g' = \sin \theta_0 g_1 + \cos \theta_0 g_2 \)

Using polar coordinates we get
\[
E \sin(g') \sin(g) = \int_0^{2\pi} \int_0^{\pi} \sin \theta \cos \phi \sin \theta' \cos \phi' \cos \theta \cos \phi' d\theta d\phi
\]
\[
\frac{3}{2\pi} \int_0^{2\pi} \sin(\theta - \phi) \frac{\sin(\phi - \phi')}{2\pi} \cos \theta \cos \phi' d\phi'
\]
Definition: A mean 0 gaussian random variable is given by
\[ g = h_i g_i \]
where \( g_i \) are independent normalized gaussian variables.

Lemma: \( (g, g') = \sin \left( \frac{\pi}{2} E(\text{sgn}(g) \text{sgn}(g')) \right) \)
where \( \text{sgn} \) is the signum function.

Proof:
\[ \begin{align*}
\Theta_0 (\eta(-1) + \Theta_0 \eta(-1) \\
+ (\frac{\pi}{2} - \Theta_0) + (\frac{\pi}{2} - \Theta_0) \\
\quad = \frac{\pi - 4 \Theta_0}{4} = \frac{1}{2} - \frac{2}{\pi} \Theta_0
\end{align*} \]

Here we get \( \beta = \frac{2}{\pi} (\pi \Theta - \Theta_0) \)
\[ \sin \left( \frac{\pi}{2} - \Theta_0 \right) = - \sin \Theta_0 \quad \text{(Wrong sign)} \]
\[ ggs' = - \sin \left[ \frac{\pi}{2} \Theta - \Theta_0 \right] \]

Proof (Gothendieck's lim) Assume that \( \delta(\zeta) \leq c \),
\( c \) to be determined. Since the Schur multiplication norm gives a Banach algebra norm, we define
\[ (\text{Sm}(w) = \sum_i \frac{x_i w_i}{w_i} (-1)^k) \]
\[ \| \text{Sm} (CM) \|_{L^2} \leq 2 \frac{1 \chi_{2w_{w_1}}}{2w_{w_1}} = \sinh (c \delta(\zeta)) \]
\[ \leq \sinh(c^2) \leq 1 \]

Now we assume
\[ \text{Sm}(c_{2d}) = \langle h, f \rangle \quad \| h \| \leq 1 \quad \| f \| \leq 1 \]
In fact we may assume \( 1 \) in both cases.

The previous lemma yields

\[ c \tilde{a}_y = \frac{\pi}{2} \mathbb{E} \sum h_c \sum d_y \]

This is not exactly what we claimed. We have shown

\[ \tilde{a}_y = \frac{\pi}{2c} \int f_1(\omega) \phi_1(\omega) d\mu(\omega) \]

and

\[ \int \sup_{y \in \mathbb{C}} \sup_{\phi \in \Phi} \phi_1(\omega) d\mu(\omega) \leq \frac{\pi}{2c} \]

provided \( \mu(\mathbb{C}) = c \).

In fact this implies that

\[ |<z, w>| \leq \frac{\pi}{2c} \|w\|_1 \|z\|_1 \]

Because

\[ |\sum \tilde{a}_y w_y| = \frac{\pi}{2c} \int \sum \phi_y(\omega) \phi_y(\omega) d\mu(\omega) \]

\[ \leq \frac{\pi}{2c} \int \sup \sum \phi_y(\omega) w_y \phi_y(\omega) d\mu(\omega) \]

\[ \leq \frac{\pi}{2c} \|w\|_1 \]

But in finite dimension

\[ \pi(\omega) = \sup \langle 2, v \rangle : \pi(v) \leq \epsilon \]

\[ \leq \pi^0(\omega) \leq \frac{\pi}{2c} \]

Thus \( \mu_0 \leq \frac{\pi}{2c} \).
Lemma: Let $X$ or $Y$ be finite dimensional. Then

$$(X \mathbin{\hat{\otimes}} Y)^* = X^* \mathbin{\hat{\otimes}} Y^* \text{ isometrically}$$

Proof: Let $\psi : X \mathbin{\hat{\otimes}} Y \to \mathbb{K}$ be a linear functional. Then the map $T_\psi : X \to Y^*$ given by

$$T_\psi(x)(y) = \psi(x \mathbin{\hat{\otimes}} y)$$

satisfies

$$\|T_\psi\| = \|\psi : X \mathbin{\hat{\otimes}} Y \to \mathbb{K}\|$$

Here we obtain an isomorphism between

$$L(X, Y^*) \cong (X \mathbin{\hat{\otimes}} Y^*)^*$$

If $X$ and $Y^*$ are finite dimensional, then

$$J : X^* \mathbin{\hat{\otimes}} Y^* \to L(X, Y^*) \text{ is surjective.} \quad \Box$$

Corollary: If $X$ and $Y$ are finite dimensional, then

$$\left\| \Pi'(z) \right\| = \sup \{ \langle z, w \rangle : \|w\|_{X^* \mathbin{\hat{\otimes}} Y^*} \leq 1 \}$$

Proof

1. $$(X \mathbin{\hat{\otimes}} Y)^* = X^* \mathbin{\hat{\otimes}} Y^*$$
2. $$(X \mathbin{\hat{\otimes}} Y)^{**} = X \mathbin{\hat{\otimes}} Y.$$
Application of G's theorem: Let $T: \ell^m_p \to \ell^n_t$. Then $T = D_A D_a D_b$ such that
$$||a||_2 ||A|| ||b||_2 \leq K ||T||.$$ 

**Proof:** We have seen that $S = D_A D_b$ when $\lambda_1 \beta < \rho_2$

$$|A: \ell^m_2 \to \ell^m_2| \leq \sup \{ |\lambda (ST_2)| \mid \lambda_2 (2) \leq 1 \}$$

$$\leq \sup \{ ||h (ST_2)|| \mid \pi_2 (2) \leq \nu_0, \}$$

$$\pi_2 (2) \leq K_0 \nu_0 (2)$$

$$\leq K_0 ||S: \ell^m_2 \to \ell^m_2||$$

Because for every linear map $S: \ell^m_2 \to \ell^m_2$ and by now we have

$$\lambda (ST_2) = \lambda (S \omega T_2) = (\omega, \lambda) = \omega_0 (2)$$

and we may view $\omega_0 (2) = \lambda (ST_2)$ as a linear function on $\ell^m_2 \otimes \ell^m_2$ with $||\omega|| = ||S||$. 

$\Box$
Theorem: Let $X$ be a Banach space and $F$ be a finite dimensional subspace of $X^*$, $G$ be a finite dimensional subspace of $X^*$ and $\epsilon > 0$. Then there exists a map $u$ from $F$ to $X$ such that

i) $\|u\| < (1 + \epsilon)$

ii) $|u(f)(g) - f(g)| < \epsilon \|f\| \|g\|$ for all $f$ in $F$ and $g$ in $G$.

Lemma: Let $X$ be a finite dimensional Banach space. Then 

$$(X \otimes Y)^{**} = X \otimes Y^{**}$$

Proof: We first prove this for $X = \ell^m_\mathbb{F}$. Then

$$\ell^m_\mathbb{F}(Y) = \ell^m_\mathbb{F} \otimes \mathbb{E} Y = L(\ell^m_\mathbb{F}, Y)$$

Because for every $S : \ell^m_\mathbb{F} \to Y$

$$\|S\| = \sup_{\|x\|_\mathbb{F} = 1} \|S(x)\| \quad (\text{convexity})$$

Now $\ell^m_\mathbb{F}(Z)^{\vee} = \ell^m_\mathbb{F}(Z^{\vee})$ (easy duality).

Hence

$$\ell^m_\mathbb{F}(Y)^{\vee \vee} = [\ell^m_\mathbb{F}(Y)^{\vee}]^{\vee} = \ell^m_\mathbb{F}(Y^{\vee})$$

Since the $\otimes$ is injective, we deduce that for every subspace $G \subseteq \ell^m_\mathbb{F}$

$$(G \otimes \mathbb{E} Y)^{\vee \vee} \subseteq \ell^m_\mathbb{F}(Y^{\vee \vee}) = \ell^m_\mathbb{F} \otimes \mathbb{E} Y^{\vee \vee} \supseteq G \otimes Y^{\vee \vee}$$
Theorem: Let $X$ be a Banach space and $F$ be a finite dimensional subspace of $X^*$, $G$ be a finite dimensional subspace of $X^*$ and $\epsilon > 0$. Then there exists a map $u$ from $F$ to $X$ such that

i) $\|u\| < (1 + \epsilon)$

ii) $|u(f)(g) - f(g)| < \epsilon \|f\| \|g\|$ for all $f$ in $F$ and $g$ in $G$.

Lemma: Let $X$ be a finite dimensional Banach space. Then $(X \otimes \mathbb{F})^{**} = X \otimes \mathbb{F}^{**}$ isometric.

Now the $\epsilon$-tensor norm is a tensor norm. Every finite dimensional space $X$ is $(1 + \epsilon)$-isomorph to a subspace of $l^N_\infty$ and hence

$$\|2 \| x \otimes \mathbb{F}^{**} \leq (1 + \epsilon) \|2 \| x \otimes \mathbb{F}^{**}$$

Since $\epsilon > 0$ is arbitrary we obtain equality $\Box$

Proof (LR): The inclusion map $i : F \rightarrow Y^*$ corresponds to a tensor $2 \otimes \mathbb{F} \otimes Y^*$ of norm 1.

Since $F \otimes Y^* = (F \otimes Y)^*$ we may apply Goldstine's theorem

$$B_2^{**} = \overline{B_2}^{\sigma(Y^*, Y)}$$

Let $g \in G$, $g^* \in G^*$ be an Amaudruk basis and $x_0 \in F$, $x_0^* \in Y^*$ also be an $\eta$-
Then \( g^* \otimes x_1 \in Y^* \otimes T \). Thus by Gâteaux we find a tensor \( \frac{d}{d \epsilon} \bigg|_{\epsilon=0} \) such that

\[
\left| \langle \frac{d}{d \epsilon} \left( \xi \otimes x_1 \right), g^* \rangle - \frac{d}{d \epsilon} \xi \langle g^*, x_1 \rangle \right| \leq \epsilon
\]

Then \( \frac{d}{d \epsilon} \) corresponds to a linear map

\[
u_\epsilon : \mathcal{F} \rightarrow Y
\]

which satisfies

\[
\left| \langle \nu_\epsilon(x_1), g^* \rangle - \frac{d}{d \epsilon} \xi \langle g^*, x_1 \rangle \right| \leq \epsilon
\]

For \( x \in \mathcal{F} \) and \( g^* \in Y^* \) we get

\[
\left| \langle \nu_\epsilon(x), g^* \rangle - \langle x, g^* \rangle \right|
\]

\[
\leq \epsilon \frac{1}{\sqrt{\text{dim}(X)}} \left( \frac{\sqrt{\text{dim}(Y)}}{\epsilon} \right)
\]

\[
\leq 3 \text{ dim} \left\| x_1 \right\|	ext{ dim} \left\| x_2 \right\|	ext{ dim} \left\| g^* \right\|
\]

Since \( \epsilon > 0 \) is arbitrary, we are done.