Homework 2

Due Date: Monday September, 11.

1. (a) Let \((X_n)\) be a sequence of metric spaces such that \(d_n(x, y) \leq 2\) holds for all \(n \in \mathbb{N}, x, y \in X_n\). On \(X = \prod_n X_n\) we define
\[
d((x_n), (y_n)) = \sum_n 2^{-n}d_n(x_n, y_n)
\]
Show that \((X, d)\) is a metric space. Assume in addition that all the \(X_n\)'s are compact. Show that \(X\) is compact. (Hint: use the diagonal procedure for sequences.)

(b) Let \(X\) be a separable normed space. This means that there exists a dense countable subset of \(X\). Let \(B_X = \{x : \|x\| \leq 1\}\) be the unit ball and \(D = \{x_n : n \in \mathbb{N}\} \subset B_X\) be a dense countable set. Define \(K_n = \{z \in \mathbb{K} : |z| \leq 1\}\) for all \(n \in \mathbb{N}\). Let \(\iota : B_X^* \to \prod_n K_n\) be defined as
\[
\iota(x^*)(n) = (x^*(x_n))_{n \in \mathbb{N}}.
\]
Show that \(\iota(B_X^*)\) is closed. (Hint: We use the topology from a) and hence you can work with sequences and limit points.)

(c) Let \(K = \iota(B_X^*)\) equipped with the induced topology. Show that \(T : X \to C(K), T(x)(\iota(x^*)) = x^*(x)\) is well-defined and satisfies
\[
\|T(x)\| = \|x\|.
\]

2. (a) Let \(X\) be a real vector space and \(K \subset X\) be a convex set such that 0 is an algebraic interior point. This means for every \(x \in X\) there exists a \(\lambda_x > 0\) such \(\lambda_x x \in K\). Show that
\[
q_K(x) = \inf\{\lambda > 0 : \frac{x}{\lambda} \in K\}
\]
is a convex function on \(X\).

(b) Let \(x_0 \in X\) such that \(x_0 \notin K\). Show that there exists a linear map \(l : X \to \mathbb{R}\) such that \(l(x_0) > 1\) and
\[
x \in K \quad \Rightarrow \quad l(x) < 1
\]
for all \(x \in K\). (Hint: Consider the subspace \(Y = \mathbb{R}x_0\).
(c) Let $K_1$ and $K_2 \subset X$ be two convex sets such that $K_1 \cap K_2 = \emptyset$. Assume that $x_1 \in K_1$ is such that such that $0$ is an algebraic interior point of $K_1 - x$ and $K_2$ is not empty. Show that there is a linear map $l : X \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that

$$l(x) \leq \alpha \leq l(y)$$

holds for all $x \in K_1$ and $y \in K_2$. (Hint: Consider $x_2 \in K_2$ and $K = (K_1 - x_1) - (K_2 - x_2) = \{x - x_1 - y - x_2 : x \in K_1, y \in K_2\}$. Then $0$ is an algebraic interior point and $x_2 - x_1 \notin K$).